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Edited for the Council by Capt. J. LAURENCE PRITCHARD, Hon. Fellow

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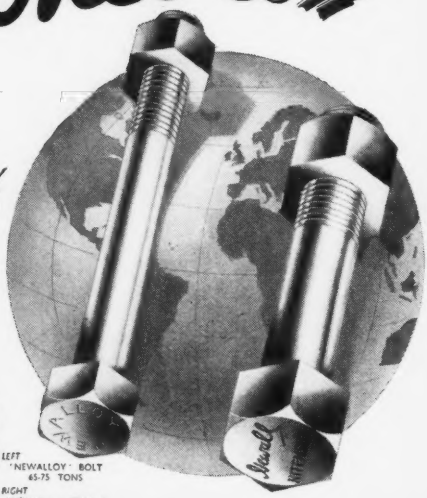

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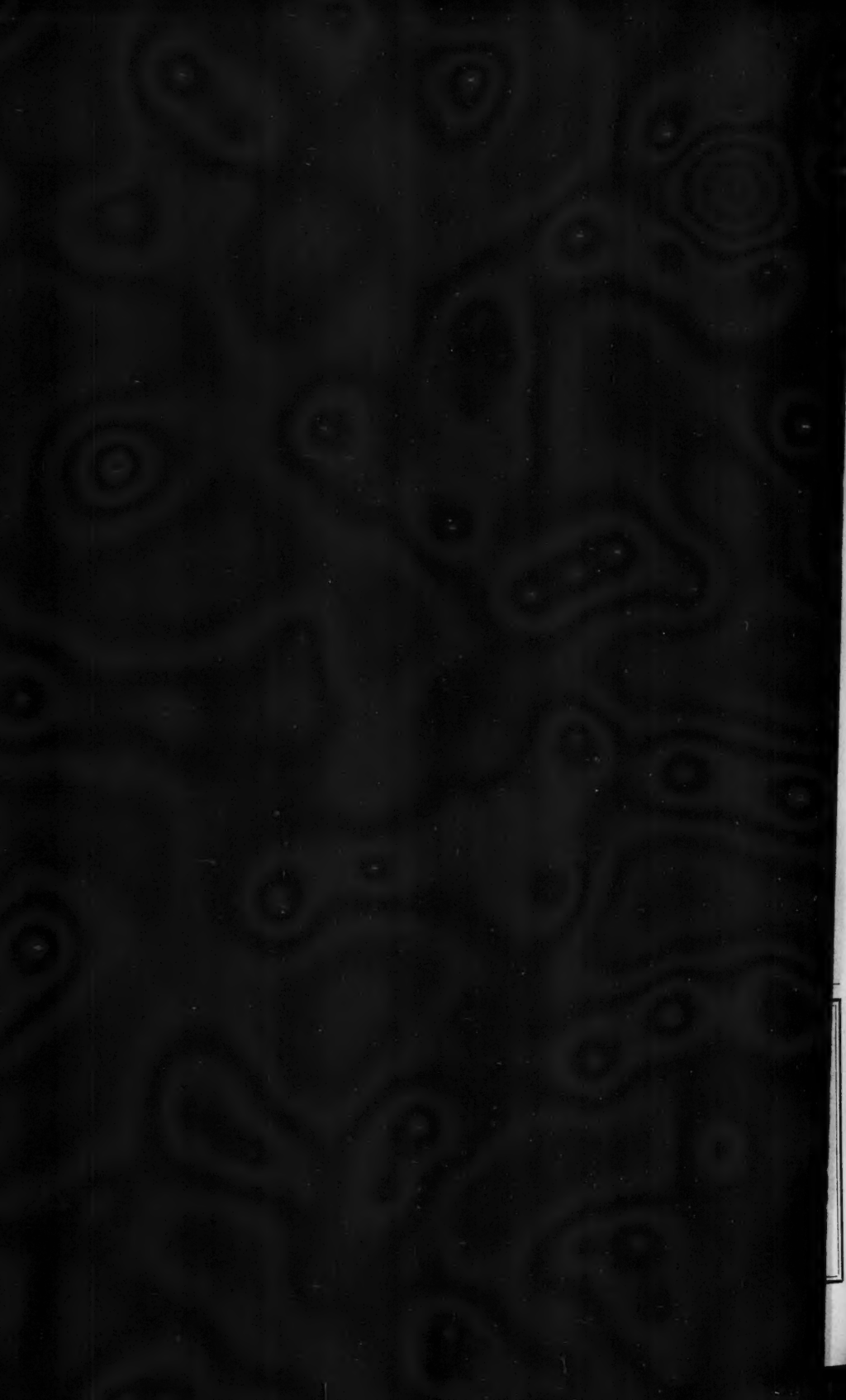

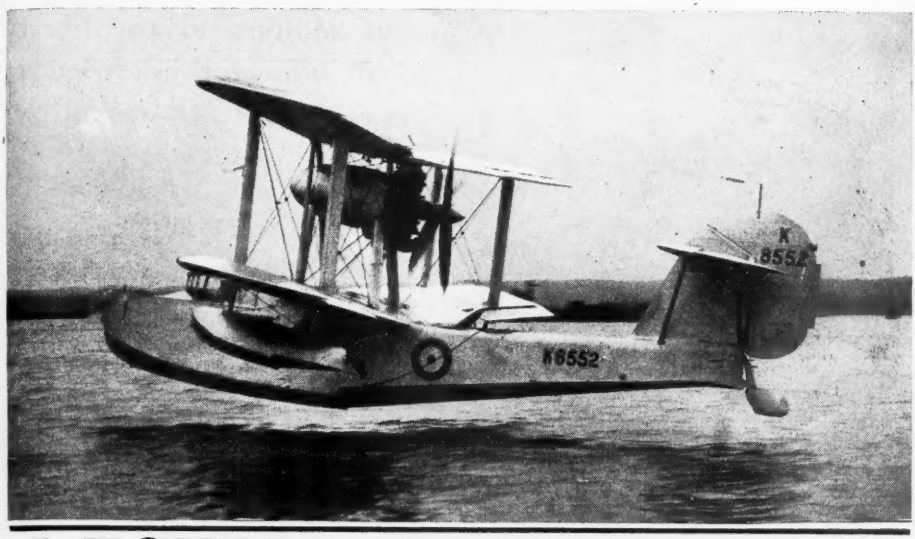

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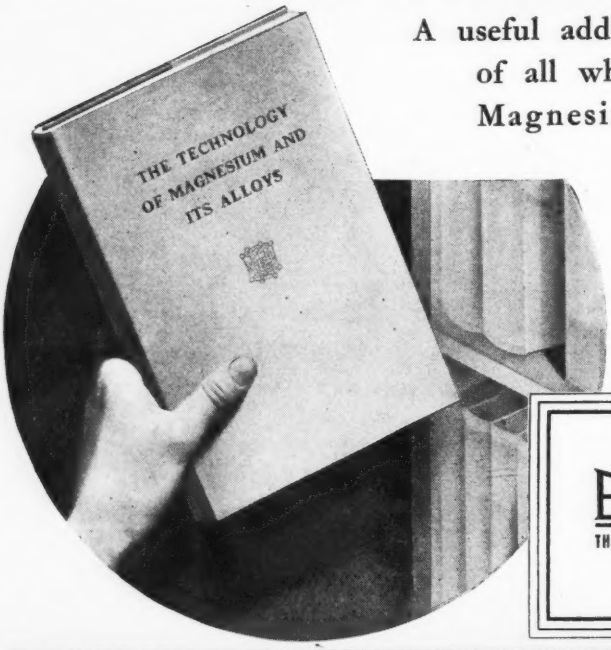

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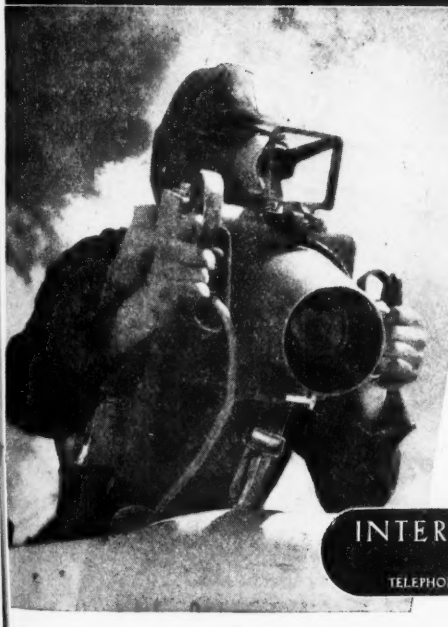

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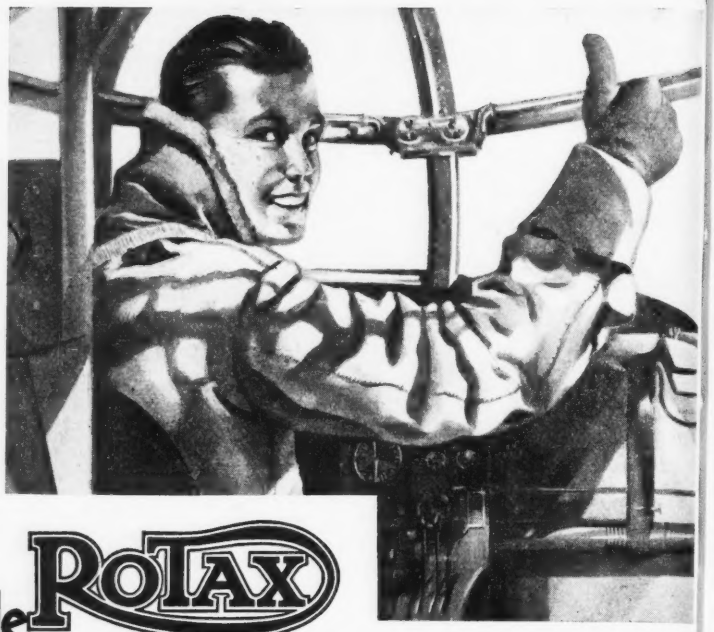

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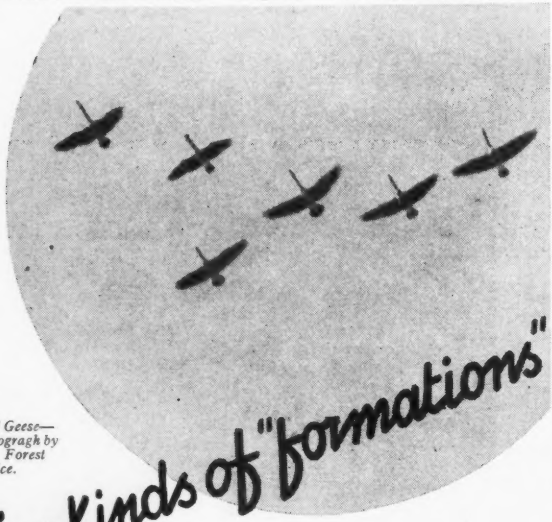

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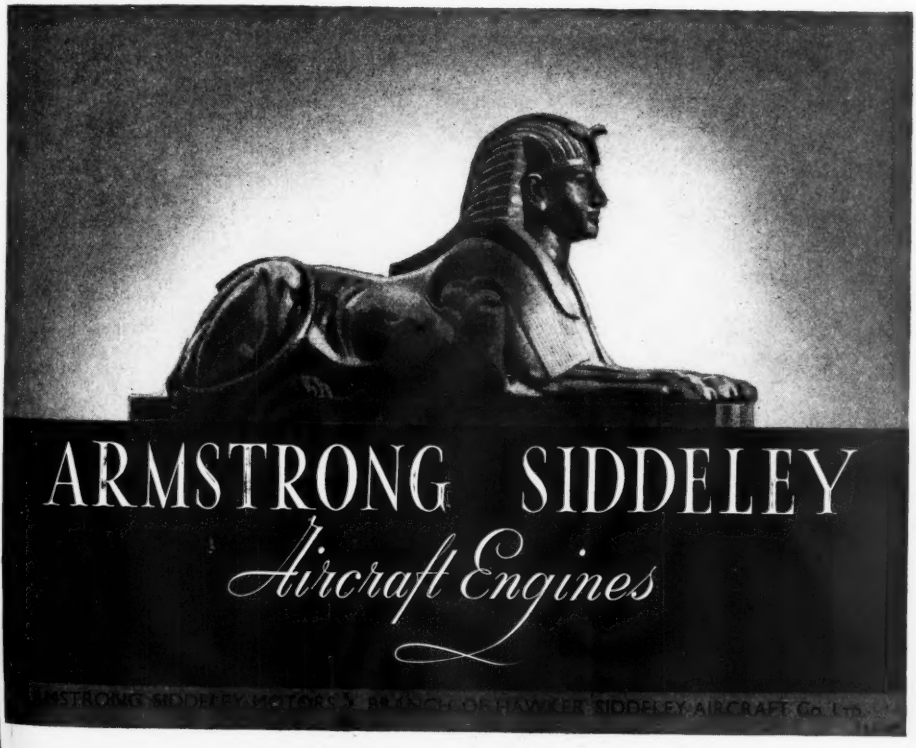
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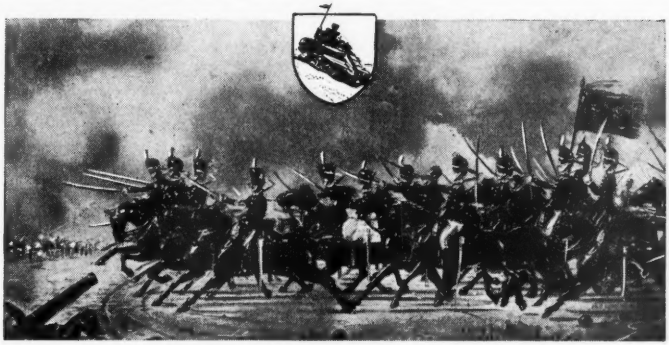

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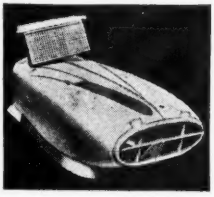

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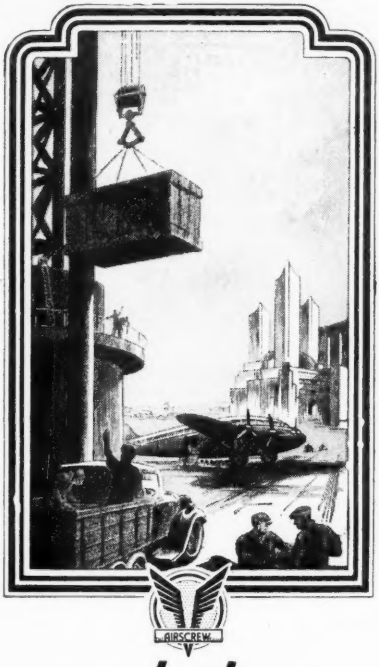

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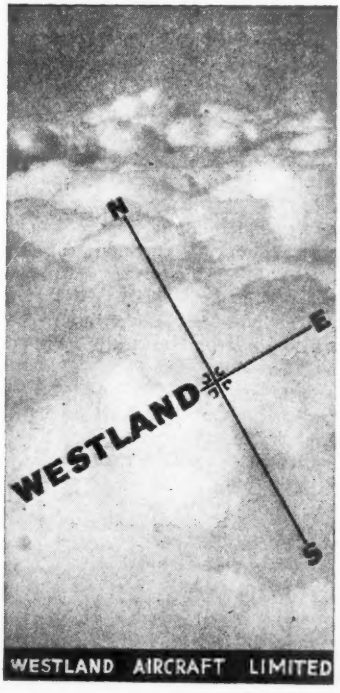

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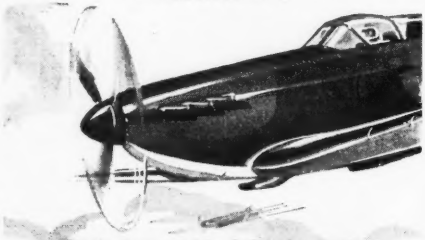

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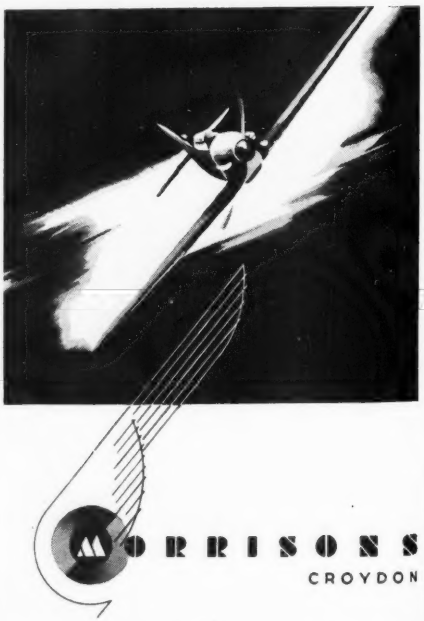

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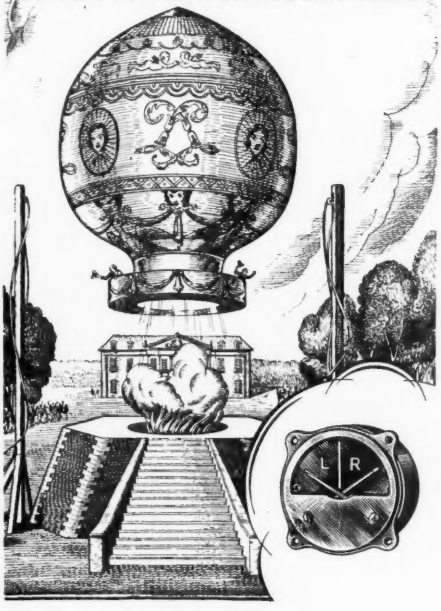

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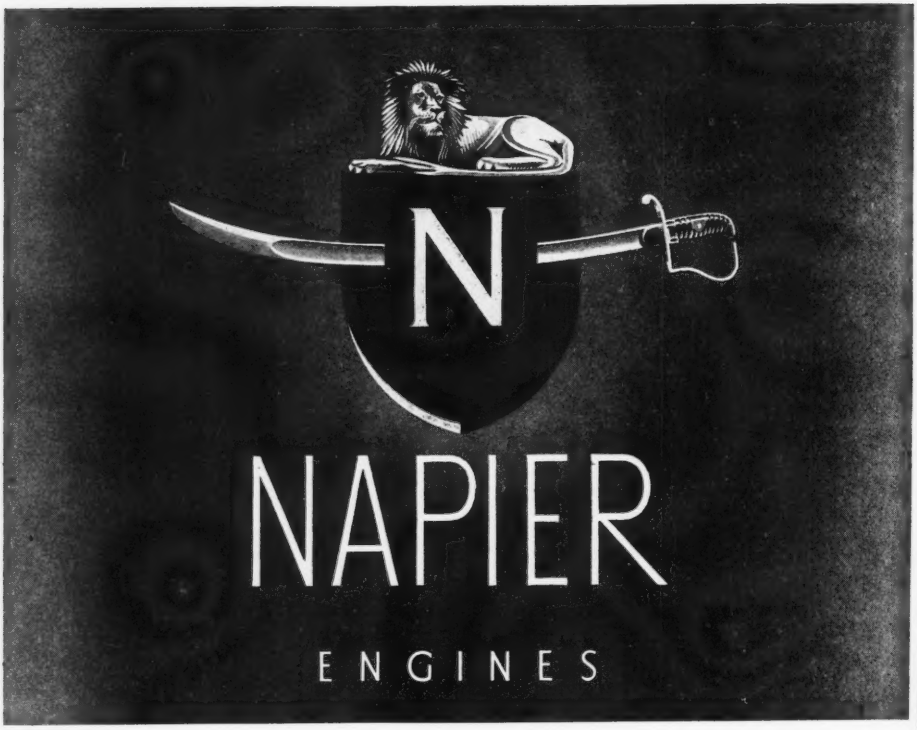
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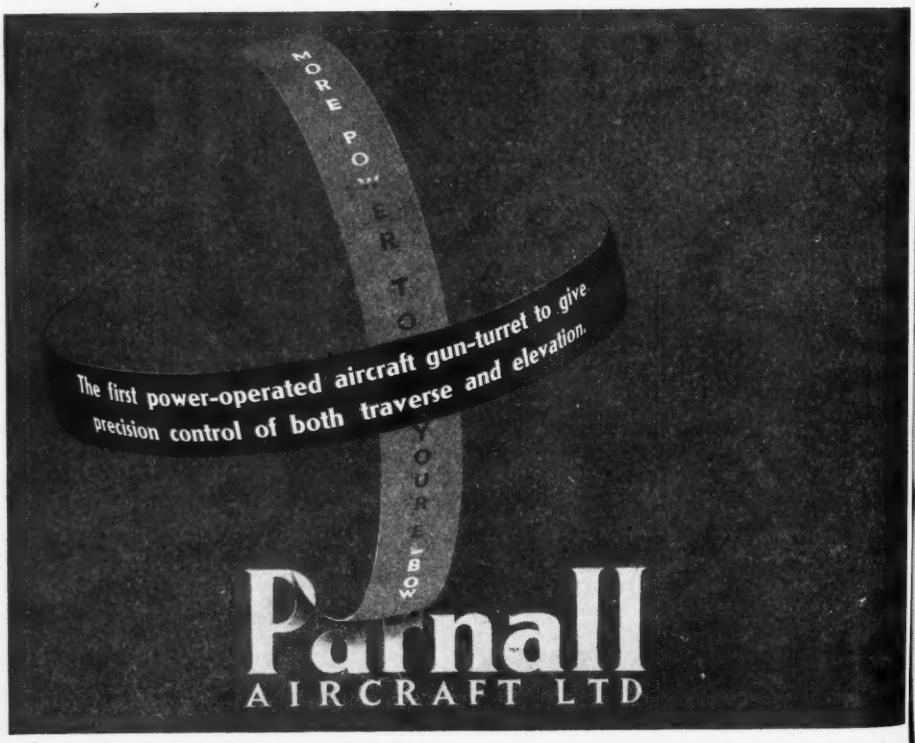
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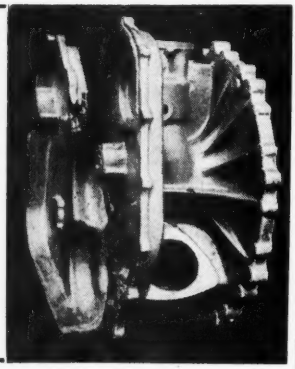

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THE ROYAL AERONAUTICAL SOCIETY

WITH WHICH IS INCORPORATED THE INSTITUTION OF AERONAUTICAL ENGINEERS

MONTHLY NOTICES

FEBRUARY, 1943.

Annual General Meeting.

Notice is hereby given that the Annual General Meeting of the Royal Aeronautical Society, with which is incorporated the Institution of Aeronautical Engineers, will be held on Saturday, March 27th, at 3.0 p.m. in the offices of the Society, 4, Hamilton Place, London, W.1.

AGENDA.

1. To read the Notice convening the Meeting.
2. To receive and deliberate upon the Report of the Council on the state of the Society and the Balance Sheets of Aerial Science Limited and Aeronautical Trusts Limited for the year ended December 31st, 1942.
3. To receive the nominations for Council for the years 1943-1945.
4. To announce the list of Fellows elected by the Council in accordance with Rule 4.
5. To elect the Auditors for the ensuing year for Aerial Science Limited and Aeronautical Trusts Limited.
6. Any other business.

By Order of the Council,

J. LAURENCE PRITCHARD,

Secretary.

Light refreshments will be served after the Meeting.

Council Election.

Attention is drawn to the following Rules for election of Council. Nominations for Council should be received by March 6th, 1943. Nomination forms may be obtained from the Secretary. Attention is drawn to possible delays in the post at the present time.

RULE 17.—One half of the Council (excluding the President and the immediate Past-President if a member of Council) shall retire annually. The members who shall retire shall be those longest in office, except as provided in Rule 77. Retiring members of Council who have served for two terms in succession (four years) shall not be eligible for re-election until the next annual election, when they will be eligible.

RULE 72.—The composition of the Council shall be as follows:—

(*a*) Not less than sixteen shall be elected from the technical or scientific grades of the R.Ae.S., that is to say, from among the Fellows, Associate Fellows, Associates and Graduates.

(*b*) Of these sixteen, four at least shall be Fellows.

RULE 73.—Nominations of candidates for election to the Council must be received by the Secretary not less than twenty-one days before the Annual General Meeting, with an intimation in writing by the Candidates that they are willing to serve. Nominations must be signed by one supporter and two seconders, who must be each entitled to vote in the R.Ae.S.

New Year Honours.

The following members were included in the New Year Honours for January, 1943. The Council wish to congratulate them on their well deserved Honours:—

K.B.E.

Acting Air Marshal J. E. A. Baldwin, C.B., C.B.E., D.S.O., O.B.E., A.F.R.Ae.S., A.D.C., R.A.F.

C.B.E.

Group Capt. P. W. S. Bulman, M.C., A.F.C., F.R.Ae.S.

C.B.E. (Military Division).

Acting Air Cmdre. K. L. Boswell, C.B.E., A.F.R.Ae.S.

O.B.E.

W/Cmdr. E. A. Gibson, A.M.Inst.C.E., A.F.R.Ae.S.
E. T. Jones, A.R.Ae.S.
Captain A. C. P. Johnstone, A.R.Ae.S.

M.B.E.

F. C. Bray, A.R.Ae.S.
Flt. Lt. J. Healey, A.R.Ae.S.
D. A. G. Jones, A.F.R.Ae.S.
Flt. Capt. A. F. Lingard, A.M.Inst.C.E., A.F.R.Ae.S.
W. A. Needs, A.R.Ae.S.

A.F.C.

Squadron Leader R. Alexander, A.F.R.Ae.S.

Associate Fellowship Examination, December, 1942.

The following candidates were successful in the December, 1942, Associate Fellowship examinations:—

C. A. Beard	Theory of Internal Combustion Engines (First Place).
	Applied Mathematics.
K. F. Brading ...	Design (Aero Engines).
	Theory of Machines (First Place).
	Applied Mathematics.
A. Herbert-Caesari ...	Aerodynamics.
	Design (Aircraft).
	Applied Mathematics.
A. F. Carr	Pure Mathematics.
M. J. G. Carson ...	Applied Mathematics.
E. B. Carter	Aerodynamics.
V. J. Chalwin ...	Aircraft Materials.
P. K. Cowan	Design (Aircraft).
	Pure Mathematics.
P. A. Curry	Design (Aero Engines).
P. J. Detmold ...	Aircraft Materials (First Place).
R. Q. E. Eden ...	Design (Aero Engines).
	Theory of Machines.
	Applied Mathematics.

M. Ellins	Design (Aircraft).
W. R. Farr	Design (Aero Engines).
	Theory of Machines.
	Applied Mathematics.
G. L. Gunstone ...	Design (Aero Engines)—First Place.
	Theory of Machines.
	Applied Mathematics.
I. A. King	Applied Mathematics.
R. A. McGill	Strength of Aeronautical Materials (First Place).
G. R. Matheson ...	Theory of Machines.
F. T. Milligan ...	Design (Aero Engines).
	Theory of Machines.
	Applied Mathematics.
B. Moss	Applied Mathematics.
W. A. E. Puntis ...	Pure Mathematics.
L. G. St. Leger ...	Pure Mathematics.
J. Smith	Pure Mathematics (First Place).
	Theory of Machines.
A. Summers	Strength of Aeronautical Materials.
	Aircraft Materials.
	Applied Mathematics.
W. Thompson ...	Aircraft Materials.
Miss V. G. Warren ...	Aerodynamics.
	Pure Mathematics.
A. N. Watson ...	Applied Mathematics.
R. Watts	Theory of Internal Combustion Engines.
	Theory of Machines.
	Applied Mathematics.
T. H. B. Weller ...	Design (Aircraft).
R. D. Williams ...	Design (Aircraft)—First Place.
V. Willis	Theory of Internal Combustion Engines.
	Applied Mathematics (First Place).
A. J. Willmore ...	Aircraft Instruments—Theory and Design.

Baden-Powell Memorial Prize.

The Baden-Powell Memorial Prize has been awarded to Mr. G. L. Gunstone, who was the best Student in the December, 1942, Associate Fellowship Examinations.

Associate Fellowship Examination, May, 1943.

Candidates for the May, 1943, Associate Fellowship Examination are reminded that entries must be in by March 31st, 1943.

Graduates' and Students' Section.

Members of the Graduates' and Students' Section are invited to the following lectures arranged by the Graduates' and Students' Section of the Institution of Automobile Engineers, 12, Hobart Place, S.W.1:—

Sunday, February 14th, 1943, at 2.30 p.m. Paper by Mr. E. S. Grillo on "Drawing Office Practice."

Sunday, March 14th, 1943, at 3.0 p.m. Paper by Mr. J. H. Millward on "The Gearbox."

Sunday, April 11th, 1943, at 3.0 p.m. Informal Talk by Mr. L. Mantell on "The Practical Aspects of Thermal Efficiency."

Sunday, May 16th, 1943, at 3.0 p.m. Informal Talk by Mr. D. H. Bramley on "Works Organisation."

Election of Members.

The following members were recently elected:—

Associate Fellows.—Eric Thomas Charles Frost (from Student), Alfred Henry Higgs (from Graduate), Herbert Frederick Leonard Jenkins, Louis William Rosenthal.

Associates.—John Sheppard Cumming, Shirley Donald Dickins (from Graduate), John Kenneth McKenzie.

Graduates.—Douglas Edward Ackland, Hanus Bloch (from Student), Norman Alexander John Harry, Jack Joseph Molins.

Student.—Henry Lambert Garbutt Sunley.

Journal Binding.

The following is a list of revised prices for binding the Society's Journals:—

1942 Volume	7s. 6d.
1941 Volume, including Library Catalogue	8s. 6d.
1941 Volume, excluding Library Catalogue	7s. 6d.
Library catalogue separately bound	6s. 0d.
Previous Volumes of Journal	8s. 6d.

Journals for binding should be sent direct to the Lewes Press, Limited, Friars Walk, Lewes, Sussex, and the remittance to this office.

Advertisements.

The fact that goods made of raw materials in short supply owing to war conditions, are advertised in the Journal, should not be taken as an indication that they are necessarily available for export.

Annual Subscriptions.

Members are reminded that their subscriptions are due on January 1st, 1943. The following are the current rates:—

	Home. £ s. d.	Abroad. £ s. d.
Founder Members	2 2 0	2 2 0
Fellows	4 4 0	3 3 0
Associate Fellows	3 3 0	2 2 0
Associates	*2 2 0	*2 2 0
Graduates (age 21-25)	2 2 0	2 2 0
,, (age 26-28)	2 12 6	2 12 6
Students	1 1 0	1 1 0
Companions	2 2 0	2 2 0

* £1 1s. 0d. without JOURNAL.

All those members who have relinquished their Journal have been notified of the reduction in their subscriptions due on January 1st, 1943.

Additions to the Library.

Pamphlets in italics with location reference following in brackets.
Books marked * or ** may not be taken out on loan.

*B.a.273.—*Ministry of Aircraft Production Official Reports. Enemy Aircraft Focke-Wulf F.W. 190 A-3. (13 photos attached). Examination of B.M.W. 801/A.1 Engine (in Dornier 217/E.1 Twin-Engined Bomber). (48 Figs. and Photos attached.)* (Y.3.iii, 11 and 12.)

B.f.85.—*How Fast do Birds Fly? By Frank W. Lane. (Article in "Country Life," Nov. 7th, 1941.)* (Y.10.b.7.)

BB.b.102.—Flugzeugleichtmetallbau. Matthiesen and Co., Berlin. 1938.

*D.a.C.19.—*Civil Aviation Statistics for Canada. June, July, August and September, 1942. Dominion Bureau of Statistics, Ottawa.*

D.b.182.—*Wings for Transportation. (Recent Developments in Air Transportation Equipment.) By T. P. Wright. Smithsonian Institution, Washington, U.S.A. (Reprint No. 3,654.)* 1942. (Y.30.ii.58.)

E.b.93.—*On the Elastic Stability of a Rectangular Plate when subjected to a Variable Edge Thrust. By D. M. A. Leggett. (Reprint from Proceedings of the Cambridge Philosophical Society, Vol. 31, Aug., 1935.) Cambridge University Press.* 1935. (Y.8.a.7a.)

*EE.h.7 and 8.—Gas Turbines and Jet Propulsion for Aircraft. By G. Geoffrey Smith. Flight Publishing Co. 1943. 3/6. (Two copies.)

G.e.A.85.—*Guide to the Properties of Wrought Aluminium Alloys. Wrought Light Alloys Development Association. (Information Bulletin No. 2.)* 1942.

*I.e.24.—Table of Circular and Hyperbolic Sines and Cosines. Federal Works Agency, New York. 1939. 20/-.

L.b.40.—"Flight" Handbook. (3rd Ed.) Flight Publishing Co. 1943. 6/-.

L.c.24.—Advanced Flying By Captain Norman Macmillan. Allen and Unwin. 1943. 2/6.

L.d.84.—Air Navigation. British Empire Edition. (2nd Ed.) By P. H. V. Weems. McGraw Hill. 1942. 35/-.

L.d.85.—Astronomical Navigation. By W. M. Smart. Longmans, Green. 1942. 5/- (2nd copy.)

L.e.10.—An Introduction to the Study of Map Projection. By James Mainwaring. Macmillan and Co. 1943. 5/-.

M.a.25.—*Elementary Electricity for Radio Students. By W. E. Flood. Longmans, Green.* 1942. 1/-. (PM.1.a.5.)

M.b.47.—*The Morse Code: Learning and Practice. By R. G. Shackel. (2nd Ed.) Longmans, Green.* 1943. 1/-. (PM.2.c.7.)

N.a.76.—*The New Frontiers in the Atom. By Ernest O. Lawrence. Smithsonian Institution, Washington, U.S.A. (Reprint No. 3,674.)* 1942. (Y.30.ii.56.)

N.a.77.—*Mathematics and the Sciences. By J. W. Lasley. Smithsonian Institution, Washington, U.S.A. (Reprint No. 3,656.)* 1942. (Y.30.ii.57.)

*P.c.38.—Abridgments of Patent Specifications: Class 4—Aeronautics. (1899-1904.) Patent Office.

**R.b.235.—*Letter from Mr. W. Le Feuvre, President of the Balloon Club, electing Mr. Percival Spencer to membership of the Club.* 1888.

**R.f.82.—*Aeronautical Medals and Tokens. By Dr. F. John Poynton. Typescript.* 1942.

R.f.83 and 84.—*The* 1914 *Tests of the Langley "Aerodrome." By C. G. Abbot. Smithsonian Institution, Washington, U.S.A. (Miscellaneous Collections, Vol. 103, No 8.)* 1942. (Y.30.i) (Two copies.)

S.d.80.—Sea Power. By "T.124." Jonathan Cape. 1940.

S.d.81.—*Italy's Prophet of Air War Havoc. (General Douhet.) Stencil of B.B.C. European Service Broadcast, December 8th,* 1942. (Y.17.c.D.)

S.e.52.—Coastal Command. Air Ministry Official Publication. H.M.S.O. 1943. 2/-.

S.e.53.—Coastal Command at War. By Sqd. Ldr. "Tom Dudley Gordon." Jarrolds. 1943. 7/6.

*V.5/47.—Proceedings of the Institution of Mechanical Engineers, Vol. 147. (Jan.-June, 1942.)

*V.7/25.—Transactions of the Institution of Naval Architects. Vol. 84. 1942.

TT.a.40.—Seven Pilots. By Charles Graves. Hutchinson and Co. 1943. 8/6.

J. LAURENCE PRITCHARD, *Secretary and Editor.*

STRESSES IN SPACE-CURVED RINGS REINFORCING THE EDGES OF CUT-OUTS IN MONOCOQUE FUSELAGES.

By N. J. Hoff, Ph.D., A.F.R.Ae.S., A.F.I.Ae.S.*

1. Introduction.

The object of this paper is to contribute one step toward the knowledge of the stresses in the neighbourhood of openings in monocoque aircraft fuselages by developing a procedure for the calculation of the stresses in spaced-curved rings used for reinforcing the edges of openings. The general appearance of such an opening is shown in the perspective view of Fig. 1. The procedure presented here presupposes a knowledge of the forces which act upon the ring. These forces depend upon the relative distortions of the reinforcing ring and of the rest of the fuselage, and the exact determination of their magnitude under different conditions of loading of the fuselage presents great difficulties.

Such a rigorous calculation of the forces is not attempted in this paper. It is rather suggested to make the assumption that the elastic properties of the ring are identical with those of the cut-out portion. On this assumption the loading of the ring consists of the stresses which would prevail in the fuselage at the location of the ring, had no opening been provided. In reality the distortions of the reinforcing ring are, as a rule, greater than those of the corresponding portion of the undisturbed fuselage, and consequently the forces acting upon the ring are usually smaller than those calculated on the assumption suggested. In general, therefore, the assumption is on the safe side from the standpoint of the necessary strength of the ring.

Characteristic features of the procedure suggested are the use of two systems of co-ordinates, one fixed to the fuselage, the other travelling along the centre line of the space-curved bar; the determination of all geometric and mechanical properties of the bar as functions of the length of its centre line; the use of vectors for the representation of directions, displacements, rotations, forces, and moments; and the consideration of the effect upon the moments of a non-uniform warping of the cross section of the space-curved bar.

Although the calculations according to the procedure suggested are basically simple, they are rather laborious. For this reason a detailed investigation is carried out in order to determine the effect of different orientations of the principal axes of inertia of the cross sections, of different values of the ratios of flexural and torsional rigidities, and of a non-uniform warping of the cross sections. From these investigations conclusions are drawn as to when it is permissible to use simplified procedures.

After the major part of the calculations presented here was finished, the writer's attention was called to a paper by K. Marguerre. (Ref. 3.) In it a complete analytical solution of the moment distribution in one particular space-curved ring is given for some important loading conditions. In cases when the geometric and mechanical properties of the space-curved ring resemble closely enough those of the ring investigated by Marguerre, it is of obvious advantage to make use of his results rather than to carry out the calculations suggested in the present paper. The magnitude of possible deviations from Marguerre's results is, therefore, investigated in some detail.

* The Polytechnic Institute of Brooklyn, New York. The major part of the paper is taken from a dissertation submitted to Stanford University, California, in partial fulfilment of the requirements for the degree of Ph.D. The author acknowledges his indebtedness to Professor S. Timoshenko for the criticism, advice, and suggestions received from him.

In the Appendix the theory of the torsion of thin-walled open sections is presented. The theory was first given by Wagner (Ref. 7) and again discussed by Kappus (Ref. 2) in papers written in German, and published in English translation by the N.A.C.A. Although these N.A.C.A. publications are easily available, their lack of clarity makes a new detailed derivation appear useful.

2. Geometric and Mechanical Properties of the Space-Curved Bar.

The shape of a space-curved bar reinforcing the edges of an opening in a monocoque fuselage is seldom, if ever, known analytically. It is usually given by two or three normal projections in a layout drawing. If one establishes a fixed system of rectangular co-ordinates x, y, z, corresponding values of these co-ordinates can easily be determined for a number of points of the centre line of the bar. From the differences of these co-ordinates for adjacent points the length of an element of the arc of the centre line of the bar,

$$\Delta s = (\Delta x^2 + \Delta y^2 + \Delta z^2)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (1)$$

can be determined with any required degree of accuracy by spacing the points sufficiently closely. With a reference point $s=0$ established on the centre line of the bar, the distance (measured along the centre line) of any point of the centre line from the reference point can be obtained by adding up the elementary lengths Δs. Thus the connection between the parameter s and the co-ordinates x, y, z, is known, and may be presented in tabular form or by three graphical plots of the functions $x=x(s)$, $y=y(s)$, and $z=z(s)$, respectively.

In the following calculations use is made of the unit tangent vector T. This is defined as a vector of unit length having the direction of the tangent to the curve and pointing toward increasing values of the parameter s. Its three components t_x, t_y, t_z in the direction of the fixed axes of co-ordinates are given by the equations

$$t_x = x'(s) \quad t_y = y'(s) \quad t_z = z'(s) \quad . \quad . \quad . \quad . \quad (2)$$

where the dash $'$ denotes a differentiation with respect to s. If the functions $x(s)$, $y(s)$, and $z(s)$ are not known analytically, the components of the unit tangent vector may be obtained by graphical differentiation. For instance, the value of t_x at any point σ is equal to the slope of the curve $x=x(s)$ at the point $x=\sigma$. Thus the values of the three components of the unit tangent vector can readily be determined and tabulated.

In addition, the components of two more unit vectors are determined and tabulated. These are the vectors P_1 and P_2, defined as vectors of unit length having the directions of the two principal axes of inertia of the section. Their sense is so chosen that the three vectors T, P_1, P_2 correspond to a right-hand system. The directions of P_1 and P_2 can be taken from the layout drawings. If the shape of the surface of the monocoque fuselage is known analytically, the directions of P_1 and P_2 can often be calculated upon the assumptions discussed in Sec. 5.

In the procedure outlined in the present paper two systems of co-ordinates are used simultaneously. In the fixed system the direction of the y-axis is taken as the direction of the axis of the horizontally disposed fuselage. The axes x and z are in a plane perpendicular to y, x is horizontal, z is vertical, and the positive sense is determined so that the axes correspond to a right-hand system. The travelling system is defined by the three mutually perpendicular unit vectors T, P_1, and P_2. The angles between the co-ordinate axes of the travelling system and those of the fixed system are, in general, different at each point of the centre line of the space-curved bar.

When the cross-sectional dimensions of the space-curved bar are large as compared with the radius of curvature of the centre line of the bar, the stress distribution due to bending is not linear over the cross section. In such a case two other vectors may be chosen in preference to P_1 and P_2 to establish the travelling system of vectors in conjunction with the tangent vector. These two

new vectors are the unit principal normal vector and the unit binormal vector of the centre line of the bar. The advantage of such a choice is due to the fact that any very small element of a space curve may be considered a plane curve, the plane of which passes through the tangent vector and the principal normal vector. This plane is called the osculating plane. If the moment acting in any section of the space-curved bar is resolved into the directions of the principal normal and binormal, the distortions of the element of the bar may be calculated with the aid of the theory of bending of plane curved bars as presented in text-books on strength of materials. Although one is seldom compelled to make use of such a refinement in the calculations, the pertinent formulæ of differential geometry are given here:—

The components of the unit principal normal vector P are

$$p_x=\rho x''(s)\quad p_y=\rho y''(s)\quad p_z=\rho z''(s) \qquad . \quad . \quad . \quad (3)$$

where ρ is the radius of curvature of the centre line of the space-curved bar. $(1/\rho)$ is given by the equation

$$(1/\rho)=(x''^2+y''^2+z''^2)^{\frac{1}{2}} \qquad . \quad . \quad . \quad . \quad (4)$$

Since the binormal vector B is a vector of unit length perpendicular to both the tangent vector and the principal normal vector, it can best be both represented and computed as the vector product of T and P, that is

$$B=\begin{vmatrix} i & j & k \\ t_x & t_y & t_z \\ p_x & p_y & p_z \end{vmatrix} \qquad . \quad . \quad . \quad . \quad (5)$$

where i, j, k are unit vectors parallel to the co-ordinates x, y, z. In all the above formulæ each dash denotes one differentiation with respect to the parameter s.

Obviously, all the above formulæ may be used for determining the unit vectors P and B graphically, if the centre line is not known as an analytic function of s. Unfortunately, however, the accuracy of two consecutive graphical differentiations is not too good.

As mentioned in the Introduction, all distortions of the bar are represented by vectors. A rotation vector is defined as a vector the length of which represents (to some convenient scale) the angle of rotation, the direction of which is parallel to the axis of rotation, and the sense of which, as shown by the arrow, corresponds to the direction of the advance of a right-hand screw which undergoes the same rotation. A moment vector is defined in the usual way, its sense is also taken as corresponding to the right-hand screw rule.

3. The Space-Curved Ring of Solid Section.

The space-curved ring corresponding to a reinforcement of the edges of the cut-out in a monocoque fuselage as shown in Fig. 1 is six times statically indeterminate. If a complete cut is made through it at any point, the unknown stresses in the cut may be replaced by a direct force (in the direction of T), two shear forces (*e.g.*, in the directions of P_1 and P_2), and by three moments the vectors of which again may be taken parallel to T, P_1, and P_2, respectively. The first of these corresponds to the twisting moment, and the second and the third to the two bending moments about the two principal axes of inertia of the section, respectively. These three forces and three moments may be chosen for the six unknowns of the problem.

In the simple structure of the cut ring it is possible to determine the forces and moments acting in every section under any known external loads. Similarly, the forces and moments in every section due to the six unknowns can be determined. Six linear equations can be obtained then for the determination of the six unknowns if one makes use of the conditions of continuity. These require the vanishing of the six components of the resultant relative displacement of the two sides of the cut. The six components are the three relative displacements measured in the direction of the vectors T, P_1 and P_2, respectively, and the three

relative rotations about them. The relative displacements can be calculated from integrals extended over the total length L of the centre line of the space-curved bar. The integrals can always be solved by numerical or graphical methods.

In the case of curved bars the cross-sectional dimensions of which are small as compared to their length, the deformations caused by direct forces and by shear forces are, as a rule, much smaller than those caused by bending and twisting moments. It is common practice, therefore, to calculate only the latter and completely disregard the former. This practice will be followed in the present paper. Consideration of the distortions due to direct force and shear adds materially to the computational work, but otherwise presents no difficulties.

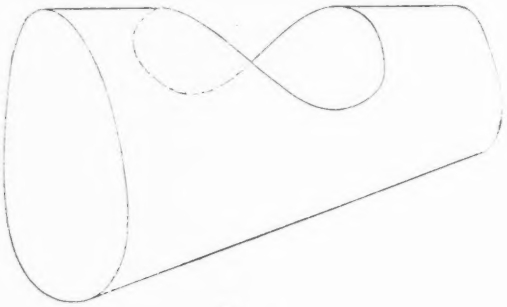

FIG. 1.

Perspective view of a cut-out in a monocoque fuselage.

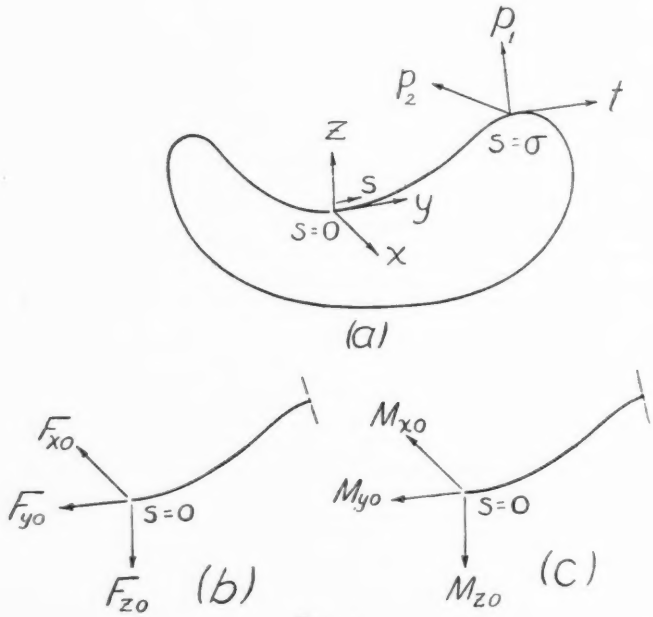

FIG. 2.

Centre line of a space-curved bar reinforcing the edge of a cut-out.

The equations for the calculation of the unknowns may be developed in the following way. In Fig. 2 let the internal moments in a section σ be called positive, if their vectors point in the positive directions of the co-ordinates on the side of the section that belongs to the portion of the bar extending from $s=0$ to $s=\sigma$. The unknown force components F_{x0}, F_{y0}, F_{z0}, and the unknown moment components M_{x0}, M_{y0}, M_{z0} at $s=0$ are drawn in directions opposite to the positive x-, y-, z- directions, since they are acting on the side of the cut opposite to that defined above. It is convenient to calculate the moments caused by the external forces acting upon the bar (not shown in the figure) with respect to the fixed system of co-ordinates x, y, z rather than with respect to the travelling system t, p_1, p_2. If these external moment components are identified by the subscript e, one may give the resultant moments acting in any section in the form :—

$$\left.\begin{aligned} M_x &= M_{xe} + M_{x0} + F_{y0}z - F_{z0}y \\ M_y &= M_{ye} + M_{y0} + F_{z0}x - F_{x0}z \\ M_z &= M_{ze} + M_{z0} + F_{x0}y - F_{y0}x \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

Before proceeding to the calculation of the distortions, one must resolve these moments in the directions of the travelling system of co-ordinates. It is known from vector calculus that the component of a vector in the direction of a unit vector is the scalar product of the two vectors. Hence the component of the moment M (given in Eqs. 6 by its three components M_x, M_y, M_z) in the direction of the tangent to the centre line of the bar is the scalar product of M and the unit tangent vector T :—

$$M_t = M \,.\, T = M_x t_x + M_y t_y + M_z t_z \quad . \quad . \quad . \quad . \quad (7a)$$

Similarly, the moments in the directions of the two principal axes of inertia of the section are :—

$$M_{p1} = M \,.\, P_1 = M_x p_{1x} + M_y p_{1y} + M_z p_{1z} \quad . \quad . \quad . \quad (7b)$$

$$M_{p2} = M \,.\, P_2 = M_x p_{2x} + M_y p_{2y} + M_z p_{2z} \quad . \quad . \quad . \quad (7c)$$

The distortions of an element of the bar of a length ds can be given by the three angles of rotation $d\beta_t$, $d\beta_{p1}$, and $d\beta_{p2}$,

$$\left.\begin{aligned} d\beta_t &= M_t ds/GC \\ d\beta_{p1} &= M_{p1} ds/EI_1 \\ d\beta_{p2} &= M_{p2} ds/EI_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where GC is the torsional rigidity of the bar according to the Saint-Venant theory, I_1 and I_2 are the two principal moments of inertia of the section, and E is Young's modulus of the material. For the calculation of the relative distortions of the two sides of the section at $s=0$ the rotation $d\beta$ (as given in Eqs. 8 by its three components $d\beta_t$, $d\beta_{p1}$, and $d\beta_{p2}$) must be resolved into the x-, y-, z- directions. Since the direction of the rotation vector $d\beta_t$ is the same as that of the unit vector T, and since the x-component of T is t_x, the x-component of $d\beta_t$ is the product $t_x d\beta_t$. Dealing with the other components in a similar way, one obtains

$$\left.\begin{aligned} d\beta_x &= t_x d\beta_t + p_{1x} d\beta_{p1} + p_{2x} d\beta_{p2} \\ d\beta_y &= t_y d\beta_t + p_{1y} d\beta_{p1} + p_{2y} d\beta_{p2} \\ d\beta_z &= t_z d\beta_t + p_{1z} d\beta_{p1} + p_{2z} d\beta_{p2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

The relative rotations of the two sides of the cut at $s=0$ can now be calculated by integrating the elementary rotations over the closed centre line of the space-curved bar. The integrals so obtained must vanish because of the continuity of the bar. Thus one obtains the three equations

$$\left.\begin{aligned} \phi d\beta_x &= 0 \\ \phi d\beta_y &= 0 \\ \phi d\beta_z &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The relative vertical displacement dw at $s=0$ due to a rotation $d\beta$ is

$$dw = x d\beta_y - y d\beta_x.$$

For reasons of continuity the integral of this expression must also vanish when extended over the closed centre line of the bar. From the requirements of a

vanishing of the relative displacements in the x- and y- directions, one obtains two more equations. Altogether one has

$$\left.\begin{aligned} \phi x d\beta_y - \phi y d\beta_x = 0 \\ \phi y d\beta_z - \phi z d\beta_y = 0 \\ \phi z d\beta_x - \phi x d\beta_z = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The integrals of the six equations Eqs. 10 and 11 can be calculated numerically, or graphically. All of them contain, in general, the six unknowns. Thus six linear equations are obtained which can always be solved for the six unknowns they contain. Substituting the values of the unknowns into Eqs. 7, one obtains the twisting and bending moments as functions of the parameter s, and thus the problem is solved.

The procedure explained above, though always possible, is rather laborious. Fortunately, most space-curved bars used in aircraft are symmetrical with respect to one or more axes. When this is the case, the properties of symmetry may be used to decrease the number of the unknowns.

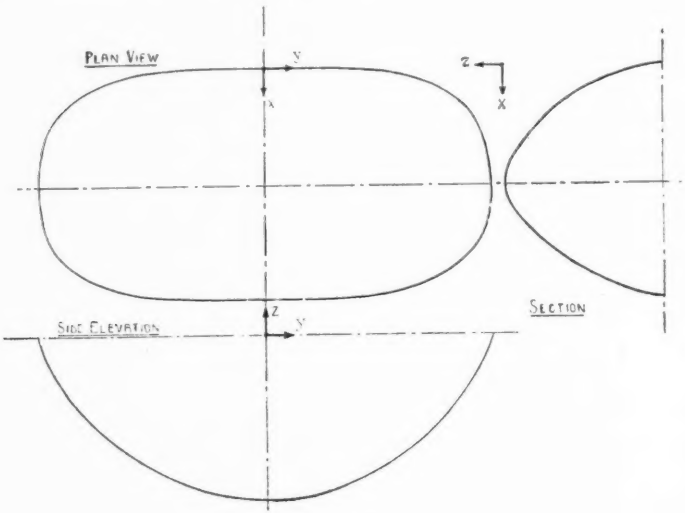

FIG. 3.

Orthogonal projections of the centre line of the space-curved bar reinforcing the edge of the cut-out shown in Fig. 1.

It should be noted that the above calculations are valid only if Hooke's law is valid, and the distortions are small as compared with the cross-sectional dimensions. Moreover, the radius of curvature of the centre line of the bar must be large as compared with the cross-sectional dimensions. This last requirement, however, need not be fulfilled if the two principal axes of inertia of every cross section coincide with the principal normal, and the binormal of the curve, respectively, and the moments of inertia are replaced by their equivalents in the theory of plane-curved bars. It is further assumed that the shape of the cross section does not change under the loads (*e.g.*, due to flattening or buckling).

4. Calculation of the Moment Distribution in a Doubly Symmetrical Ring.

As an example, the moment distribution in the doubly symmetrical space-curved bar, the centre line of which is shown in Fig. 3, is calculated. The curve may

be considered as representing the edges of a cut-out in the top portion of a smaller size monocoque fuselage. The calculations are carried out analytically rather than graphically in order to facilitate the investigation of the effects upon the moment distribution of changes in the values of the three rigidities GC, EI_{p1}, and EI_{p2}.

The loading of the ring consists of two equal and opposite forces of a magnitude $2P$ each acting parallel to the y-axis at the topmost points of the curve. In Fig. 4 one-half of the ring is shown together with the loading. The location of the origin of the fixed system of co-ordinates differs from that in Sec. 3. The effect of this change upon the equations of Sec. 3 is readily found. It is assumed arbitrarily that the directions of the principal axes of inertia of every section coincide with the principal normal, and with the binormal, respectively, of the curve.

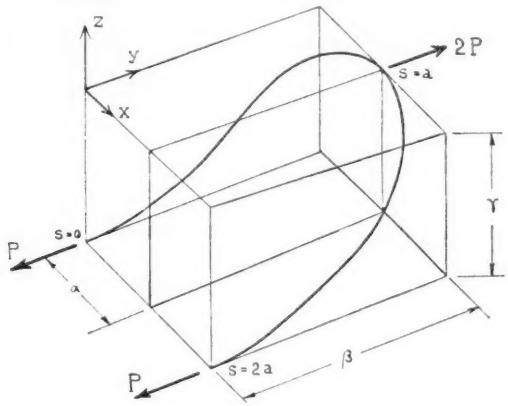

FIG. 4.

Axonometric view of one-half of the space-curved bar of Fig. 3.

The co-ordinates of the curve are given by the equations:—

$$\left.\begin{aligned} x &= (a/3)\,(s/a)^3 \\ y &= (a/3)\,[3\,(s/a) - (s/a)^3] \\ z &= -(a/3)\sqrt{2}\,[1-(s/a)^2]^{3/2} \end{aligned}\right\} \quad . \quad . \quad . \quad (12)$$

The equations are valid if

$$0 \leq (s/a) \leq 1$$

where a is the length of one-quarter of the complete space-curved ring. The complete ring is obtained by mirroring the above-defined curve in planes parallel to the x- and z-, and to the y- and z-axes, respectively.

The three components of the unit tangent vector T of the curve were calculated from Eqs. 2. They are

$$\left.\begin{aligned} t_x &= (s/a)^2 \\ t_y &= 1-(s/a)^2 \\ t_z &= \sqrt{2}\,(s/a)\,[1-(s/a)^2]^{\frac{1}{2}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (13)$$

The principal normal vector P can be calculated from Eqs. 3. Its components are

$$\left.\begin{aligned} p_x &= \sqrt{2}\,(s/a)\,[1-(s/a)^2]^{\frac{1}{2}} \\ p_y &= -\sqrt{2}\,(s/a)\,[1-(s/a)^2]^{\frac{1}{2}} \\ p_z &= 1-2\,(s/a)^2 \end{aligned}\right\} \quad . \quad . \quad . \quad (14)$$

The length of the radius of curvature is calculated with the aid of Eq. 4. It is

$$\rho = (a/\sqrt{2})\,[1-(s/a)^2]^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (15)$$

The binormal vector B is calculated from Eqs. 5. Its components are

$$\left.\begin{aligned} b_x &= 1-(s/a)^2 \\ b_y &= (s/a)^2 \\ b_z &= -\sqrt{2}\,(s/a)\,[1-(s/a)^2]^{\frac{1}{2}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (16)$$

Due to the double symmetry of both ring and loading, stresses and distortions are identical in the four quarters of the ring. It suffices, therefore, to consider only one quarter. If a cut is made both at $s=0$ and $s=a$, in Fig. 4, the following force and moment components must vanish in the cuts because of the symmetry :—

At $s=0$: the shear force parallel to x, the shear force parallel to z, and the torque (moment vector parallel to y);

At $s=a$: the shear force parallel to z; and the torque (moment vector parallel to x).

Moreover, and still because of the symmetry, the tensile force at $s=0$ (force parallel to y) is P, and the shear force parallel to y at $s=a$ is also P. The values of some more force and moment components follow then from the requirements of static equilibrium :

At $s=0$: the moment vector parallel to x is $-P\gamma$.

At $s=a$: the tensile force (parallel to x) is zero, and the moment vector parallel to y is zero.

Thus the only two unknowns in the cuts are the moment vectors parallel to z. The two, however, are interdependent. If one denotes the unknown moment at $s=0$ by M_{z0}, the moment at $s=a$ becomes $M_{z0}-Pa$. With this notation one obtains for the moments at any point of the centre line of the ring :—

$$\left.\begin{aligned} M_x &= Pz \\ M_y &= 0 \\ M_z &= M_{z0}-Px \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

These equations correspond to Eqs. 6 of the preceding section. They contain a single unknown, namely M_{z0}, which must be calculated from the condition of continuity. Substituting the expressions of Eqs. 13-17 into Eqs. 7 and remembering that the vectors P_1 and P_2 are coincident with the vectors P and B, respectively, one obtains :—

$$\left.\begin{aligned} M_t &= -(Pa/3)\sqrt{2}\,(s/a)^2\,[1-(s/a)^2]^{\frac{1}{2}} + M_{z0}\sqrt{2}\,(s/a)\,[1-(s/a)^2]^{\frac{1}{2}} \\ M_p &= -(Pa/3)\,(s/a)\,[2-3\,(s/a)^2] + M_{z0}\,[1-2\,(s/a)^2] \\ M_b &= -(Pa/3)\sqrt{2}\,[1-2\,(s/a)^2]\,[1-(s/a)^2]^{\frac{1}{2}} - M_{z0}\sqrt{2}\,(s/a)\,[1-(s/a)^2]^{\frac{1}{2}} \end{aligned}\right\} \quad (18)$$

The values of these twisting and bending moments were calculated in terms of Pa and M_{z0}. They are plotted in Fig. 5, where the subscript P refers to the former part of the moment, M to the latter. If one now calculates the relative distortions of the sections $s=0$ and $s=a$, instead of extending the integration of the elementary distortions over the total length of the closed curve, the requirements set forth in Eqs. 10 and 11 must be revised. It is easy, however, to establish end conditions for the rotations, and to use them for the determination of the unknown moment M_{z0}. The actual values of the rotations $\Delta\beta$ must fulfill the following requirements of symmetry :—

At $s=0$:	$\Delta\beta_x=0$	$\Delta\beta_y=$any value	$\Delta\beta_z=0$
At $s=a$:	$\Delta\beta_x=$any value	$\Delta\beta_y=0$	$\Delta\beta_z=0$

Accordingly, the rotation about the z-axis must vanish at both ends of the quarter-curve. Consequently

$$\int_0^a d\beta_z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Eq. 19 now replaces Eqs. 10 and 11. The definite integral on its left side can be calculated if one substitutes Eqs. 18 into Eqs. 8 (remembering that the subscripts p and b replace the subscripts p_1 and p_2, respectively), and the values

so obtained into the last of Eqs. 9. After integration between the given limits Eq. 19 becomes:—

$$(1/GC)\,[(-1/18)\,Pa^2+(4/15)\,M_{zo}a]+(1/EI_p)\,[(-1/12)\,Pa^2+(7/15)\,M_{zo}a]$$
$$+(1/EI_b)\,[(1/18)\,Pa^2+(4/15)\,M_{zo}a]=0 \quad . \quad . \quad . \quad (19a)$$

The solution of Eq. 19*a* may be written in the form

$$M_{zo}=f_M Pa \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where

$$f_M=(5/12)\,(2+3g-2h)/(4+7g+4h) \quad . \quad . \quad . \quad (20a)$$

and the parameters g and h are defined as

$$g=(GC/EI_p) \text{ and } h=(GC/EI_b) \quad . \quad . \quad . \quad (20b)$$

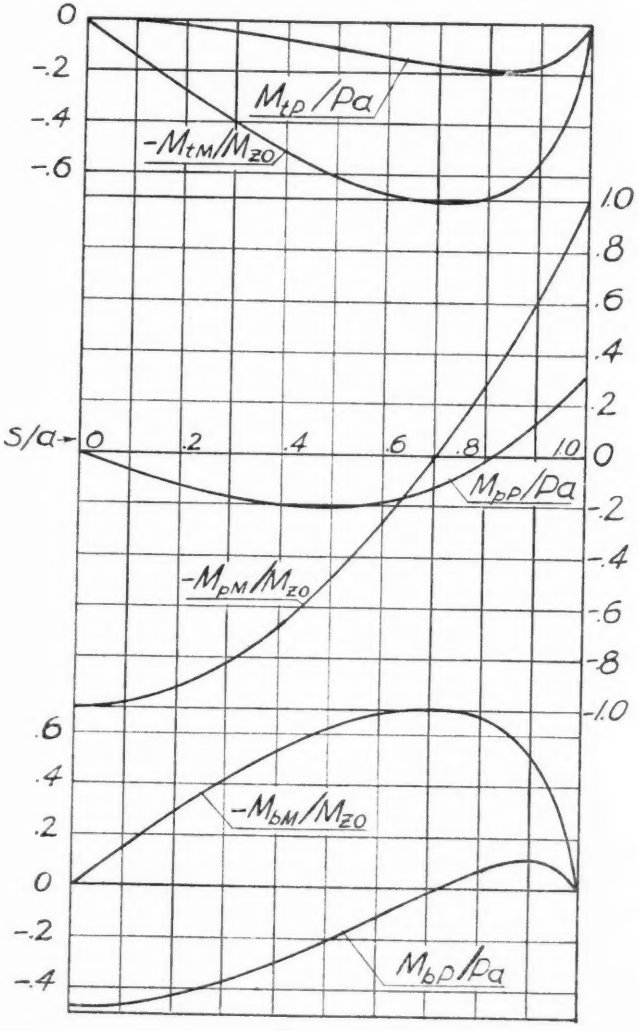

FIG. 5.

Moment distribution along the space-curved bar.

Calculated values of f_M are plotted against g and h in Fig. 6. This figure permits an analysis of the effect of the torsional and flexural rigidities upon the redundant moment M_{zo}. This moment changes most rapidly with a varying EI_b, most slowly with a varying EI_p. A variable EI_b corresponds to a vertical line in Fig. 6, a variable EI_p to a horizontal line, and a variable torsional rigidity GC to a straight slanting line through the origin, if in each case the other two rigidities are held constant. Fig. 6 shows that the redundant moment is greater, the greater the flexural rigidity EI_b about the binormal, and the smaller the torsional rigidity GC. The flexural rigidity EI_p about the principal normal is of lesser importance.

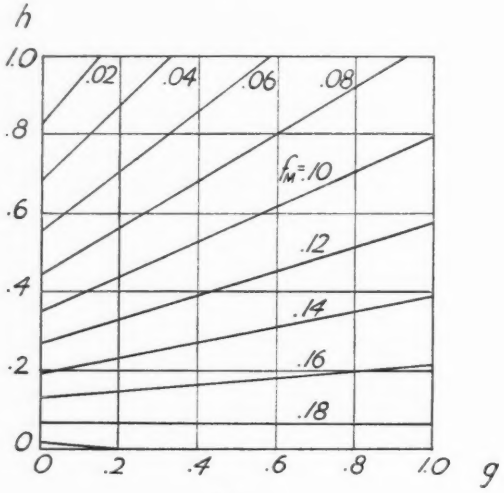

FIG. 6.

Plot of f_M of Eq. 20 against $g=CG/EI_p$ and $h=GC/EI_b$. The principal normal and the binormal are the principal axes of inertia.

If the moment diagrams of Fig. 5 are also taken into consideration, it may be seen that the effect of the redundant moment M_{zo} is to reduce the torsion of the bar. With a small value of GC and a high value of EI_b the resultant torque M_t almost vanishes, but at the same time the maximum values of the bending moments are not increased. The shape of the moment diagrams, however, is greatly affected. On the other hand, the resultant moment diagrams are very little affected by the redundant moment if the torsional rigidity and the ratio I_p/I_b are large.

The details of the above conclusions naturally hold true only for the curve considered here, and for the assumed orientation of the principal axes of inertia. The effect of a change in the latter is investigated in the next section. It may be stated generally, however, that if the torsional rigidity GC is small, the redundant moment tends to cause a decrease in the values of the twisting moments so that the bar carries the external loads mainly by virtue of its bending rigidities. The same conclusion was drawn by Marguerre from his investigations.

5. The Most Likely Directions of the Principal Axes of Inertia.

In many cases the section used for reinforcing the edges of a cut-out in a monocoque fuselage is intentionally given a permanent twist during the manu-

facturing process so that one of its plane surfaces is everywhere approximately tangential to the surface of the fuselage. In this way the section can easily be riveted to the covering of the fuselage. If the section is an I, or a channel, one principal axis of inertia is then coincident with the normal of the surface of the fuselage; if an angle, or a Z, the angle between the surface normal and the principal axis of inertia is constant. The direction of the surface normal, however, can be calculated if the shape of the surface is known analytically.

It is shown in differential geometry that if the co-ordinates x, y, z of the surface are known functions of two parameters u, v:—

$$\begin{aligned} x &= x\,(u, v) \\ y &= y\,(u, v) \\ z &= z\,(u, v) \end{aligned}$$

and the differential coefficients of these functions exist and can be calculated with respect to both u and v, and are represented by the symbols

$$\begin{array}{lll} x_u = \partial x/\partial u & y_u = \partial y/\partial u & z_u = \partial z/\partial u \\ x_v = \partial x/\partial v & y_v = \partial y/\partial v & z_v = \partial z/\partial v \end{array}$$

then the surface normal N is

$$N = \begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This is not a unit vector. It can be transformed into one by dividing it by its length.

If the surface of the fuselage is cylindrical, its parametric equation is easily found from the equation of the space-curved bar. If, for instance, the generatrices are parallel to the y-axis, and the centre line of the bar is given by the equations

$$x = x\,(s) \qquad y = y\,(s) \qquad z = z\,(s)$$

then the parametric equation of the surface may be written in the form

$$x = x\,(s) \qquad y = v \qquad z = z\,(s)$$

where s and v are the independent parameters. Thus the surface normal becomes

$$N = \begin{vmatrix} i & j & k \\ x'\,(s) & 0 & z'\,(s) \\ 0 & 1 & 0 \end{vmatrix}$$

Accordingly the components of the surface normal are

$$n_x = -z'\,(s) \qquad n_y = 0 \qquad n_z = x'\,(s)$$

The vector can be normalised by dividing it by its absolute magnitude. Hence, the components of the unit surface normal vector are

$$\left.\begin{aligned} n_x &= -z'\,(s)/[x'^2\,(s) + z'^2\,(s)]^{\frac{1}{2}} \\ n_y &= 0 \\ n_z &= x'\,(s)/[x'^2\,(s) + z'^2\,(s)]^{\frac{1}{2}} \end{aligned}\right\} \quad . \quad . \quad . \quad (21a)$$

If the surface normal is one principal axis of inertia, the other is perpendicular to both the surface normal and the tangent of the curve. Consequently, the vector product of the unit tangent vector and the unit surface normal vector is a unit vector R which points in the direction of the second principal axis of inertia. It is given by the equation

$$R = \begin{vmatrix} i & j & k \\ t_x & t_y & t_z \\ n_x & n_y & n_z \end{vmatrix} . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

6. The Effect of the Directions of the Principal Moments of Inertia upon the Redundant Moment.

With a view to investigating the effect of the directions of the principal moments of inertia upon the magnitude of the redundant moment M_{z0}, the calculations of Sec. 4 are repeated here on the assumption that the directions of the principal axes of inertia are those of the vectors N and R defined in the preceding section.

In the case of the cylindrical surface defined by the curve given in Sec. 4 the components of the unit surface normal vector are:—

$$\left.\begin{aligned} n_x &= -\sqrt{2}\,[1-(s/a)^2]^{\frac{1}{2}}/[2-(s/a)^2]^{\frac{1}{2}} \\ n_y &= 0 \\ n_z &= (s/a)/[2-(s/a)^2]^{\frac{1}{2}} \end{aligned}\right\} \quad . \quad . \quad . \quad (23)$$

The components of the unit vector R are:—

$$\left.\begin{aligned} r_x &= -(s/a)\,[1-(s/a)^2]/[2-(s/a)^2]^{\frac{1}{2}} \\ r_y &= (s/a)\,[2-(s/a)^2]^{\frac{1}{2}} \\ r_z &= -\sqrt{2}\,[1-(s/a)^2]^{3/2}/[2-(s/a)^2]^{\frac{1}{2}} \end{aligned}\right\} \quad . \quad . \quad (24)$$

With these values the twisting and bending moments become:—

$$\left.\begin{aligned} M_t &= -(Pa/3)\sqrt{2}\,(s/a)^2\,[1-(s/a)^2]^{\frac{1}{2}} + M_{z0}\sqrt{2}\,(s/a)\,[1-(s/a)^2]^{\frac{1}{2}} \\ M_n &= (Pa/3)\,[2-4\,(s/a)^2+(s/a)^4]/[2-(s/a)^2]^{\frac{1}{2}} \\ &\qquad + M_{z0}\,(s/a)/[2-(s/a)^2]^{\frac{1}{2}} \\ M_r &= (Pa/3)\sqrt{2}\,(s/a)\,[1-(s/a)^2]^{3/2}/[2-(s/a)^2]^{\frac{1}{2}} \\ &\qquad - M_{z0}\sqrt{2}\,[1-(s/a)^2]^{3/2}/[2-(s/a)^2]^{\frac{1}{2}} \end{aligned}\right\} \quad (25)$$

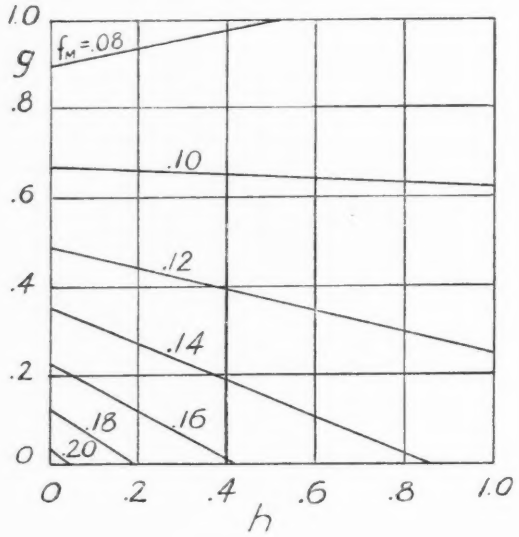

FIG. 7.

Plot of f_M of Eq. 26 against $g=GC/EI_n$ and $h=GC/EI_r$. The surface normal is one principal axis of inertia, the other is perpendicular to the surface normal, and to the tangent to the centre line of the bar.

Carrying out the calculations in the same way as in Sec. 4, one obtains

$$M_{z0} = f_M Pa \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where

$$f_M = [.0555 - .01896\,g + .04673\,h]/[.266 + .244\,g + .4854\,h] \quad . \quad (26a)$$

and the parameters g and h are the same as those defined in Eqs. 20*b*. Calculated values of f_M are plotted against g and h in Fig. 7.

Marguerre in his calculations (see Ref. 3) assumes that one of the principal axes of inertia of the section is always horizontal. The manufacturing of such a space-curved bar is comparatively easy, its attachment to the sheet covering

of the fuselage, however, is difficult. The moment distribution in the bar, the centre line of which was defined in Sec. 4, is now calculated on the assumption that one of its principal axes of inertia is horizontal.

The vector pointing in the direction of the first principal axis of inertia must then be perpendicular to both a vertical unit vector and the tangent vector. Denoting this vector by H, one has

$$H=\begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ (s/a)^2 & 1-(s/a)^2 & \sqrt{2}\,(s/a)\,[1-(s/a)^2]^{\frac{1}{2}} \end{vmatrix}$$

Consequently

$$h_x=(s/a)^2-1 \quad h_y=(s/a)^2 \quad h_z=0$$

The unit vector of the same direction has the components

$$\left.\begin{aligned} h_x&=[(s/a)^2-1]/[2\,(s/a)^4-2\,(s/a)^2+1]^{\frac{1}{2}} \\ h_y&=(s/a)^2/[2\,(s/a)^4-2\,(s/a)^2+1]^{\frac{1}{2}} \\ h_z&=0 \end{aligned}\right\} \quad . \quad . \quad (27)$$

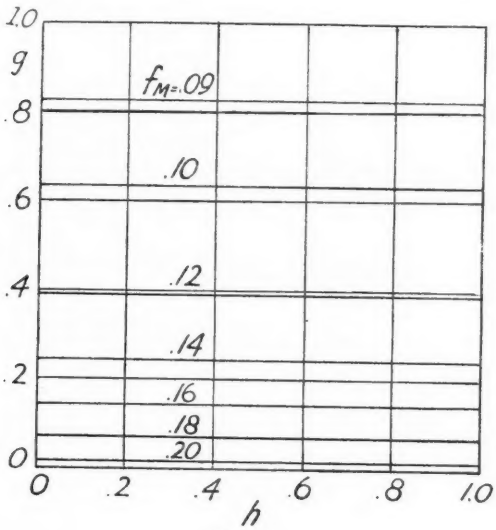

FIG. 8.

Plot of f_M of Eq. 30 against $g=GC/EI_h$ and $h=GC/EI_n$. One principal axis of inertia is horizontal, the other is perpendicular to the first principal axis, and to the tangent to the centre line of the bar.

The unit vector N having the direction of the second principal moment of inertia of the section can be calculated as the vector product of the unit vectors H and T. Its components are

$$\left.\begin{aligned} n_x&=\sqrt{2}\,(s/a)^3\,[1-(s/a)^2]^{\frac{1}{2}}/[2\,(s/a)^4-2\,(s/a)^2+1]^{\frac{1}{2}} \\ n_y&=\sqrt{2}\,[(s/a)-(s/a)^3]\,[1-(s/a)^2]^{\frac{1}{2}}/[2\,(s/a)^4-2\,(s/a)^2+1]^{\frac{1}{2}} \\ n_z&=-[2\,(s/a)^4-2\,(s/a)^2+1]^{\frac{1}{2}} \end{aligned}\right\}. \quad (28)$$

The moment distribution is given by the equations

$$\left.\begin{aligned} M_t&=-(Pa/3)\sqrt{2}\,(s/a)^2\,[1-(s/a)^2]^{\frac{1}{2}}+M_{zo}\sqrt{2}\,(s/a)\,[1-(s/a)^2]^{\frac{1}{2}} \\ M_h&=(Pa/3)\sqrt{2}\,[(s/a)^2-1]^{5/2}/[2\,(s/a)^4-2\,(s/a)^2+1]^{\frac{1}{2}} \\ M_n&=-(Pa/3)\,(s/a)^3\,[1-2\,(s/a)^2]/[2\,(s/a)^4-2\,(s/a)^2+1]^{\frac{1}{2}} \\ &\quad -M_{zo}\,[2\,(s/a)^4-2\,(s/a)^2+1]^{\frac{1}{2}} \end{aligned}\right\} \quad (29)$$

For the unknown moment the expression is obtained

$$M_{z0}=f_M Pa \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where

$$f_M=(2+g)/(9.6+26.4\,g) \quad . \quad . \quad . \quad . \quad (30a)$$

and the parameter g has the same meaning as before. It may be seen that in the present case the redundant moment is independent of h. Computed values of f_M are plotted against g and h in Fig. 8.

It should be noted that the twisting moment M is given by the same expressions in all the three cases discussed, as may be seen from a comparison of Eqs. 18, 25, and 29. The magnitude of f_M is, therefore, representative of the decrease in the values of the twisting moment due to the redundant moment M_{z0}. A comparison of the Figs. 6-8 reveals that f_M is independent of the directions of the principal axes of inertia if $g=h$. On the other hand, the differences in the value of f_M for different orientations of the principal moments of inertia are material, if one of the parameters g and h is small, the other large. Consequently the values of the twisting moment depend considerably upon the orientation of the principal axes of inertia if the bending rigidity of the space-curved bar is materially different in the two principal directions, and if the torsional rigidity GC is not too small.

7. Non-Uniform Warping.

The calculation of the redundant moment M_{z0} in Sec. 3 was carried out upon the assumption that the unit angle of twist of the curved bar was everywhere proportional to the torque, as expressed in the first of Eqs. 8. This is, however, true only if Saint-Venant's assumptions concerning simple torsion, as stated in Sec. A.1 of the Appendix, are satisfied. If the torque varies along a bar of thin-walled open section, the angle of twist has to be calculated from the differential equation

$$GC\,(d\beta_t/ds)-E\Gamma\,(d^3\beta_t/ds^3)=M_t \quad . \quad . \quad . \quad (31)$$

This equation is derived in Sec. A.5. The symbol Γ denotes the warping constant which is defined in Eq. A.28. The meanings of the other symbols were previously explained.

The seemingly inconsistent procedure of calculating the angle of twist of an element from a differential equation, while the angles of rotation due to bending are determined from the much simpler formulæ of Eqs. 8, has a simple physical reason. In bending, the well-established assumption that plane sections before bending remain plane after bending eliminates all interference between the elements of the bar during distortions. Thus the distortion of each element can be calculated individually. On the other hand, in torsion the effect of the warping of the originally plane cross sections is of primary importance and if it is not considered entirely erroneous results are obtained for both the angle of twist and the shearing stress, as was shown by Saint-Venant a century ago. Thus, if due to a variable torque, or to some restraint to warping, the warping is not uniform over the whole bar, the distortions of each element are influenced by the distortions of all the other elements.

In the case of solid, or thick-walled closed sections the calculation of the effect of non-uniform warping is much more involved. Fortunately, with such sections the effect is negligibly small, except in the immediate proximity of concentrated loads and constraints. The angle of twist can be calculated, therefore, from the first of Eqs. 8 without undue error. For solid, or hollow circular sections the first of Eqs. 8 is rigorously correct, since such sections do not warp when twisted.

The edge reinforcements most often encountered in monocoque aircraft, however, are made of thin-walled open sections for which both terms on the left side of Eq. 31 are of equal importance. Even with sections of extremely thin wall thickness, for which $E\Gamma$ may be much larger than GC, it is not advisable to

disregard the first term on the left-hand side of Eq. 31. If the term is disregarded, the equation is obviously incorrect in the case of a constant torque M_t.

Eq. 31 may be given in the form

$$K^2\beta_t' - \beta_t''' = (1/E\Gamma)\, M_t(s) \quad . \quad . \quad . \quad . \quad (31a)$$

where

$$K^2 = (GC/E\Gamma), \quad . \quad . \quad . \quad . \quad . \quad (31b)$$

each dash $'$ denotes one differentiation with respect to s, and the symbol $M_t(s)$ is used in order to denote that the torque is a function of the parameter s. The solution of the reduced equation

$$K^2\beta_{tr}' - \beta_{tr}''' = 0 \quad . \quad . \quad . \quad . \quad . \quad (32)$$

can be written as

$$\beta_{tr}' = A \cosh Ks + B \sinh Ks \quad . \quad . \quad . \quad . \quad (33a)$$

or

$$\beta_{tr}' = A_1 e^{Ks} + B_1 e^{-Ks} \quad . \quad . \quad . \quad . \quad . \quad (33b)$$

The solution of the complete equation Eq. 31a is

$$\beta_t' = \beta_{tr}' + \beta_{tp}' \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where β_{tp}' is a particular solution. This can be found by the operational method

$$(K^2 - D^2)\,\beta_{tp}' = (1/E\Gamma)\, M_t(s)$$

$$\beta_{tp}' = (1E\Gamma)\,[1/(K^2 - D^2)]\, M_t(s)$$

The latter equation may be transformed into

$$\beta_{tp}' = (1/2KE\Gamma)\,\{\,[1/(D+K)] - [1/(D-K)]\,\}\, M_t(s)$$

From this form it is clear that β_{tp}' can be obtained as the sum of the solutions of two linear differential equations of the first order. These solutions are well known. Since only the particular solutions are sought here, the following expression is obtained:—

$$\beta_{tp}' = (1/2KE\Gamma)\,\left[e^{-Ks}\int e^{Ks} M_t(s)\, ds - e^{Ks}\int e^{-Ks} M_t(s)\, ds\right] \quad . \quad (35)$$

When this equation is substituted into Eq. 31a, an identity is obtained. Hence the differential equation Eq. 31a is solved for any type of distributed torque and the determination of the angle of twist is reduced to the solution of the two integrals in Eq. 35. This can always be accomplished with the aid of numerical, or graphical methods. The two constants of integration in Eq. 33a (or in Eq. 33b) must be determined from the boundary conditions.

The redundant moment M_{zo} can now be calculated according to the procedure discussed in Sec. 3, or in the case of double symmetry by that discussed in Sec. 4. The only difference is that in the calculations the values of β_t' obtained from the differential equation Eq. 31 must, of course, be used in place of the values computed from the first of Eqs. 8.

8. Numerical Example.

In order to obtain a numerical appraisal of the effect of non-uniform warping upon M_{zo}, the example of Sec. 4 is continued here. With the arbitrary assumption that

$$K = 1/a$$

where a is again the (developed) length of one-quarter of the space-curved bar, Eq. 35 becomes

$$\beta_{tp}' = (a/2E\Gamma)\,\left[e^{-s/a}\int e^{s/a} M_t(s)\, ds - e^{s/a}\int e^{-s/a} M_t(s)\, ds\right] \quad . \quad . \quad (35a)$$

After substituting the value of $M_t(s) = M_t$ from the first of Eqs. 18 into Eq. 35a, one has to integrate the terms containing Pa, and those containing M_{zo} separately. The integration was carried out numerically. Its results for the " moment "-terms (those containing M_{zo}) are given in the table below:—

s/a =	0	.1	.2	.4	.6	.8	.9	1.0
$-(2E\Gamma/\sqrt{2}M_{zo}a^2)\,(\beta_{tp}')_M$	0	.0004	.0026	.0208	.069	.158	.216	.288

Writing the solution of the complete equation in the form

$$(\beta_t')_M = A \cosh (s/a) + B \sinh (s/a) + (\beta_{tp}')_M \quad . \quad . \quad (36)$$

one can determine the values of the constants of integration from the end conditions. Due to the double symmetry the warping must vanish at both end sections of the bar. As may be seen from the derivations of the Appendix, this is the case when $\beta_t'=0$. Hence the two end conditions give

$$\left.\begin{aligned} \beta_t'=0 \text{ when } s=0 \\ \beta_t'=0 \text{ when } s=a \end{aligned}\right\}$$

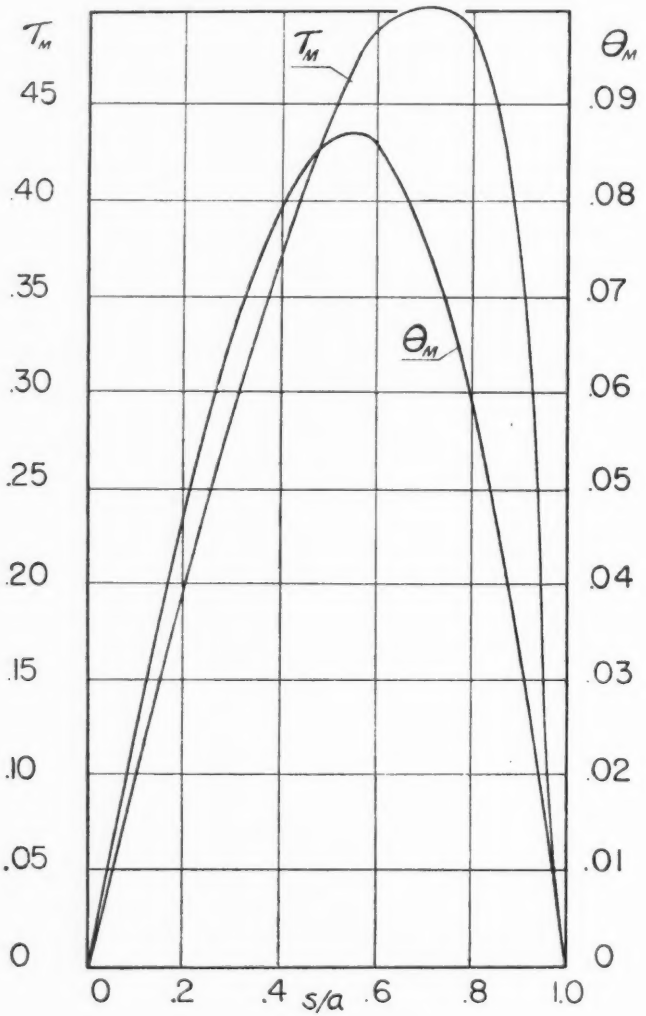

FIG. 9.

Moment distribution $\tau_M = M_{tM}/\sqrt{2}\, M_{zo}$, *and unit angle of twist distribution* θ_M (*Eq.* 38).

Since $(\beta_{tp}')_M$ is zero when $s=0$, one obtains

$$A=0$$

For $s=a$ the particular solution is

$$(\beta_{tp}')_M=-.288\sqrt{2}M_{zo}a^2/2E\Gamma$$

Hence

$$B=.288\sqrt{2}M_{zo}a^2/2E\Gamma\sinh(1)=.244\sqrt{2}M_{zo}a^2/2E\Gamma$$

Substituting these values into Eq. 36, one may write

$$(\beta_t')_M=\theta_M\sqrt{2}M_{zo}a^2/2E\Gamma \quad . \quad . \quad . \quad . \quad (38)$$

where θ_M is a numerical factor. Its values are plotted against (s/a) in Fig. 9. The figure also contains calculated values of τ_M defined as the factor by which $\sqrt{2}M_{zo}$ is multiplied in the first of Eqs. 18. Consequently the τ_M curve is proportional to the torque, and the θ_M curve to the unit angle of twist. It may be seen from Fig. 9 that the two curves greatly resemble each other. In the case of a vanishing resistance to warping their ordinates would be proportional.

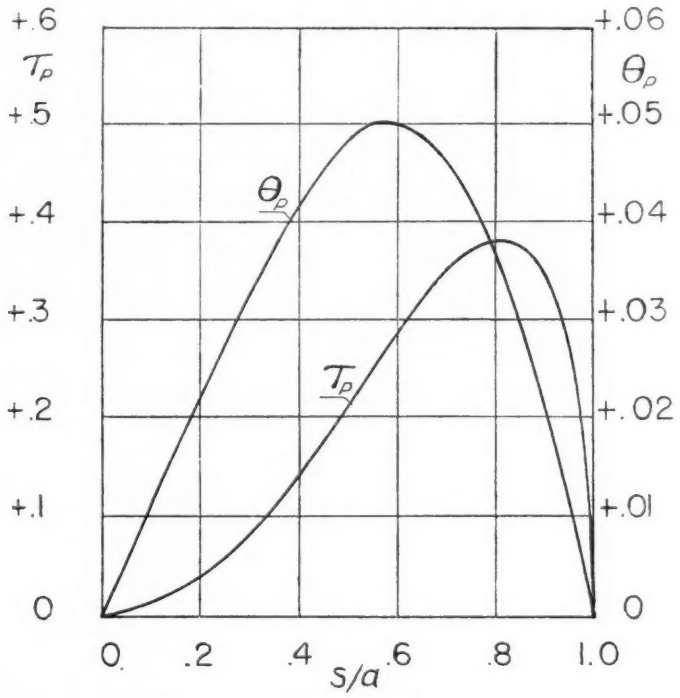

FIG. 10.

Moment distribution $\tau_P=-3M_{tP}/\sqrt{2}Pa$, *and unit angle of twist distribution* θ_P (*Eq.* 39c).

The calculation of the " force " terms (those containing Pa) is quite similar. From the end condition one obtains

$$A=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (39a)$$

$$B=-.111\ (\sqrt{2}/3)\ Pa^3/2E\Gamma \quad . \quad . \quad . \quad . \quad (39b)$$

and thus one may write

$$(\beta_t')_P=-\theta_P\sqrt{2}Pa^3/6E\Gamma \quad . \quad . \quad . \quad . \quad (39c)$$

where θ_P is a numerical factor. Its values are plotted against (s/a) in Fig. 10. The figure also contains calculated values of τ_P which is defined as the factor by which $-\sqrt{2Pa}/3$ is multiplied in the first of Eqs. 18. In Fig. 10 the shape of the θ_P and τ_P curves differs considerably; in the case of a vanishing resistance to warping their ordinates would be proportional.

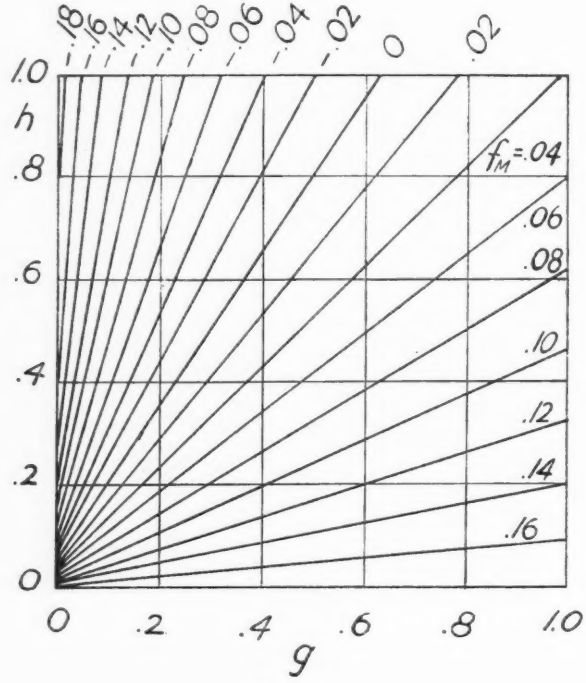

FIG. 11.

Plot of f_M of Eq. 41 against $g=GC/EI_p$, and $h=GC/EI_b$. Principal axes of inertia as in Fig. 6. Effect of non-uniform warping considered.

With the aid of the values of β_t' so obtained one can calculate M_{zo}. First the values of $(\beta_t')_M$ and $(\beta_t')_P$ must be multiplied by the corresponding values of t_z in order to obtain the z-components of the rotations due to torsion. With these products plotted against s, a diagram is obtained in which the areas below the two curves are proportional to the angles of twist due to the " moment "-terms, and the " force " terms, respectively. Carrying out the numerical calculations, one obtains

$$\int_0^a t_z\beta_t'ds = .0226\sqrt{2}M_{zo}a^3/2E\Gamma - .01275\sqrt{2}Pa^4/6E\Gamma \quad . \quad . \quad (40)$$

The values of

$$\int_0^a p_z\beta_p'ds \text{ and } \int_0^a b_z\beta_b'ds$$

remain the same as in Sec. 4. Since $a^2 = 1/K^2 = E\Gamma/GC$ one may write Eq. 40 in the form

$$\int_0^a t_z \beta_t' ds = .016\, M_{z0} a/GC - .003\, Pa^2/GC \quad . \quad . \quad . \quad (40a)$$

The numerical factors on the right-hand side of Eq. 40a are 16.6 times, and 18.5 times smaller, respectively, than the corresponding terms in Eq. 19a. This shows that the torsional stiffness of the bar was increased to such an extent by the resistance to non-uniform warping. The expression for the redundant moment is

$$M_{z0} = f_M Pa \quad . \quad . \quad . \quad . \quad . \quad (41)$$

where

$$f_M = (.045 + 1.25g - .833h)/(.24 + 7g + 4h) \quad . \quad . \quad . \quad (41a)$$

and g and h were defined in Section 4. Computed values of f_M are shown in Fig. 11. The figure differs materially from Figs. 6, 7, and 8.

9. The Effect of Non-Uniform Warping upon the Redundant Moment.

If the space-curved bar is of thin-walled open section, its rigidity in simple torsion is very much smaller than that in bending. Thus in monocoque aircraft structures the usual values of g and h are around, or below 1/100. A comparison of Figs. 6 and 11 shows, however, little difference in the value of f_M in this region. For very small values of g and h Eqs. 20a and 41a yield .208, and .188, respectively, for f_M. It seems, therefore, that the value of the redundant moment is hardly affected by the increase in the rigidity of the bar due to the effects of non-uniform warping, although the calculations of the preceding article showed an increase of the rigidity to over 16 times its original value.

The reason for this interesting phenomenon is that in the example the non-uniform warping had just about the same stiffening effect in the case of the external moment, as in that of M_{z0}. This is true of many practical problems, but not necessarily so. If the moment distribution due to the external loads differs materially from that due to the redundant moment, the stiffening effect may be different for different values of the ratio $K^2 = GC/E\Gamma$; the value of the redundant moment will then also turn out differently.

In order to get an insight into the numerical effects of different loadings upon the redundant moment, the bar shown in Fig. 12a is now investigated. The bar is rigidly fixed against torsion and warping at its left end $x = L$. At its right end, $x = 0$, it is held fixed against rotation but it is allowed to warp freely. It is loaded with a distributed torque m expressed in in. lbs. per in. according to the formula

$$m = (\mu/\sinh n) \sinh (nx/L) \quad . \quad . \quad . \quad . \quad (42)$$

As the formula shows, the maximum value of m is μ, and this value is reached at the left end. The shape of the distribution curve varies with the parameter n and is shown in Fig. 12b for three different values of the parameter.

If one chooses for the unknown the moment M_0 exerted upon the bar by the fixation at the right-hand side support, the " simple " system is a cantilever bar with its left end rigidly fixed against rotation and warping. In this the bending moment due to the distributed torque is

$$M = (\mu L/n \sinh n)\,[\cosh (nx/L) - 1] \quad . \quad . \quad . \quad (43)$$

Substituting this value into Eq. 31a, one obtains

$$K^2\beta' - \beta''' = (\mu L/E\Gamma n \sinh n)\,[\cosh (nx/L) - 1] \quad . \quad . \quad (44)$$

The end conditions are:—

$$\beta' = 0 \text{ when } x = L$$
$$\beta'' = 0 \text{ when } x = 0$$

The solution of Eq. 44 is

$$\beta' = (\mu L/GCn \sinh n)\{[1-(K^2L^2\cosh n)/(K^2L^2-n^2)](\cosh Kx)/(\cosh KL) + K^2L^2\cosh(nx/L)/(K^2L^2-n^2)-1\} \quad . \quad . \quad . \quad (45)$$

provided that

$$KL \neq n \quad . \quad . \quad . \quad . \quad . \quad . \quad (45a)$$

The relative angle of rotation β_μ of the two end sections due to the distributed torque can be obtained by integrating the right-hand side of Eq. 45 between the limits $x=0$ and $x=L$. One obtains

$$\beta_\mu = (\mu L^2/GCn \sinh n)\{[1-(K^2L^2\cosh n)/(K^2L^2-n^2)](1/KL)\tanh KL + K^2L^2(1/n)\sinh n/(K^2L^2-n^2)-1\} \quad . \quad . \quad . \quad (46)$$

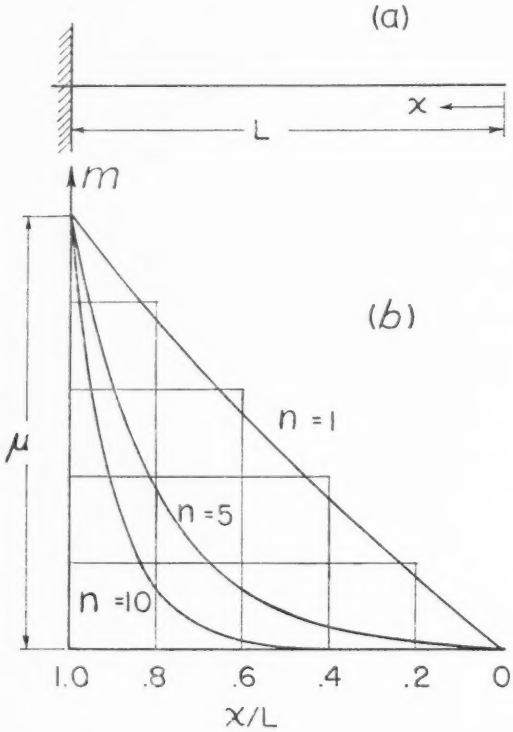

FIG. 12.

(a) Bar fixed at both ends against rotation, at left end against warping.
(b) Distribution of applied torque of Eq. 42.

In a similar way one may calculate the relative angle of rotation β_M of the two end sections of the bar due to a constant M_o. The result is

$$\beta_M = (M_oL/GC)[1-(1/KL)\tanh KL] \quad . \quad . \quad . \quad (47)$$

The redundant moment M_o may now be calculated from the condition that the relative rotation of the two fixed ends must be zero:—

$$\beta_\mu + \beta_M = 0 \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Solving for M_o one obtains

$$M_o = (\mu L/n\sinh n)\{1-[K^2L^2(1/n)\sinh n]/(K^2L^2-n^2) - [1-(K^2L^2\cosh n)/(K^2L^2-n^2)](1/KL)\tanh KL\}/\{1-(1/KL)\tanh KL\} \quad (49)$$

Next the value M_{oo} of the redundant moment is calculated for the case when the bar does not offer any resistance to warping. The unit angle twist is then proportional to the moment :—

$$\beta' = M/GC = (\mu L/GCn \sinh n) [\cosh (nx/L) - 1] \quad . \quad . \quad (50)$$

The relative angle of twist β_μ due to the external loads is

$$\beta_\mu = (\mu L^2/GCn \sinh n) [(1/n) \sinh n - 1] \quad . \quad . \quad . \quad (51)$$

The relative angle of twist β_M due to the constant moment M_{oo} is

$$\beta_M = M_{oo} L/GC \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Applying Eq. 48 and solving for M_{oo}, one obtains

$$M_{oo} = (\mu L/n \sinh n) [1 - (1/n) \sinh n] \quad . \quad . \quad . \quad (53)$$

The effect of warping may now be calculated by determining the ratio M_o/M_{oo}. This was done for the three loading conditions represented by the values 1, 5, and 10 of the parameter n. Computed values may be found in Fig. 13.

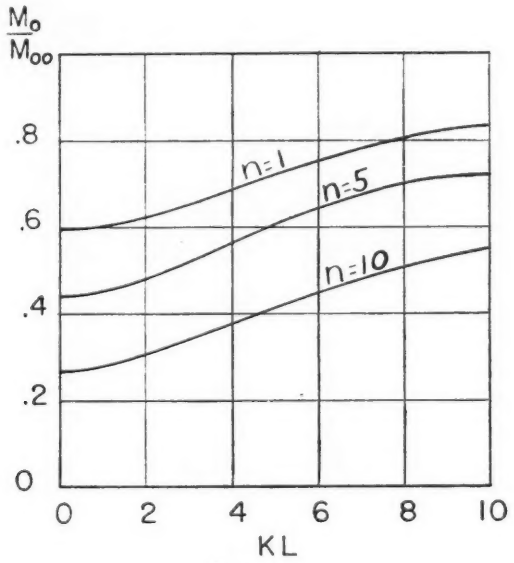

FIG. 13.

Ratio of the unknown moment M_o to the unknown moment M_{oo}, the latter calculated without consideration of the effect of non-uniform warping.

Eq. 49 is not suited to numerical calculations when KL is very small. In such cases an expansion in power series yields the simpler formula

$$M_o = (\mu L/n \sinh n) \{ 1 - (3/n^2) [\cosh n - (1/n) \sinh n] \} \quad . \quad (54)$$

It was mentioned above that Eq. 45 was not the solution of Eq. 44 when $KL = n$. In this case the solution is

$$\beta' = (\mu L/GC \sinh n) \{ (\tfrac{1}{4}) \cosh Kx - (\tfrac{1}{2}) Kx \sinh Kx - 1 - [(\tfrac{1}{4}) \cosh KL - (\tfrac{1}{2}) KL \sinh KL - 1] (\cosh Kx)/(\cosh KL) \} \quad (55)$$

Integrating the right-hand side of Eq. 55 between the limits $x = 0$ and $x = L$ and solving for M_o, one obtains

$$M_o = (\mu L/n \sinh n) \{ KL + 2KL \cosh KL - 2 \sinh KL - (\tfrac{1}{2}) \sinh 2KL \} / \{ 2KL \cosh KL [1 - (1/KL) \tanh KL] \} \quad (56)$$

Fig. 13 contains computed values of M_o/M_∞ in the region

$$0 \leq KL \leq 10$$

The usual range in monocoque aircraft structures is within this region, as is shown in Sec. 11. The figure shows that there are considerable changes in the values of M_o/M_∞ when the loading and KL are varied.

10. Calculation of the Unit Stress.

After the statically indeterminate quantities have been determined with the aid of considerations of continuity, the calculation of the stresses in the cross sections does not present major difficulties. Stresses are caused by the direct force and the shearing forces as well as by the bending and twisting moments in each section, and the determination of the maximum combined stress is rather laborious. In this section the calculation of the stresses caused by torsion in thin-walled open sections is discussed in some detail.

As shown in the Appendix (see Eq. A14, but substitute β for ϕ), the direct stress σ due to non-uniform warping is given by the equation

$$\sigma = Ew\beta_t''$$

If for β_t' an analytical expression is known, β_t'' is easily calculated by differentiation. If, however, β_t' was obtained from a numerical, or graphical integration, the differentiation cannot be carried out analytically. In such a case two procedures are possible. The easier one consists of a graphical differentiation of the $\beta_t' = \beta_t'(s)$ function. The starting point for the more accurate one is Eq. 34. If one substitutes the expressions of Eq. 33a and Eq. 35 into Eq. 34, and then differentiates with respect to s, one obtains

$$\beta_t'' = K(A \sinh Ks + B \cosh Ks) - (1/2E\Gamma)\left[e^{-Ks}\int e^{Ks} M_t(s)\, ds + e^{Ks}\int e^{-Ks} M_t(s)\, ds\right] \quad . \quad . \quad . \quad . \quad (57)$$

Since the values of the two integrals in Eq. 57 are known for a number of points along the centre line of the bar (that is for a number of values of the parameter s) from the numerical integration, and since the values of the constants A and B were also determined when β_t' was calculated, the computation of β_t'' from Eq. 57 does not involve much additional work. The value of the warping w can be calculated from Eq. A6. (See also Sec. A7.)

In order to obtain numerical values for an appraisal of the importance of the effects of non-uniform warping upon the strength of the bar, the example of Secs. 4 and 8 is continued here. For the calculation of the stresses it is necessary to choose a definite section. The choice made here is a channel 2 ins. by 1 in. by .04 in. Its sectional properties are as follows:—

$$I_p = .1067 \text{ in.}^4$$
$$C = .0853 \times 10^{-3} \text{ in.}^4$$
$$K^2 = GC/E\Gamma = .00281 \text{ in.}^{-2}$$
$$I_b = .0166 \text{ in.}^4$$
$$\Gamma = .01166 \text{ in.}^6$$

Assuming the length of one-quarter of the bar

$$a = 20 \text{ ins.}$$

one obtains

$$Ka = 1.06$$

This value is close enough to unity so that the results of the calculations of Sec. 8 may be used without appreciable error. With

$$g = GC/EI_p = .308 \times 10^{-3} \text{ and}$$
$$h = GC/EI_b = 1.97 \times 10^{-3}$$

one obtains from Eq. 42a

$$f_M = .175$$

and thus Eq. 41 becomes

$$M_{z0}=.175\,Pa \quad . \quad . \quad . \quad . \quad . \quad (58)$$

From Eq. 57 one can now calculate β_t'' making use of the numerical values computed in Sec. 8. The results may be presented in the form

$$\beta_t''=C''\sqrt{2P/2GC} \quad . \quad . \quad . \quad . \quad . \quad (59)$$

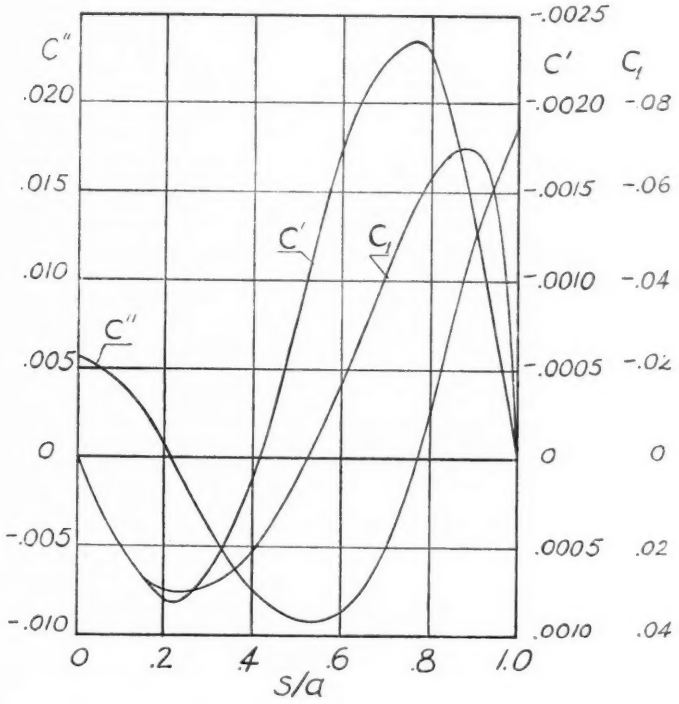

FIG. 14.

Numerical example of Sec. 10. $C_1=M_t/Pa$;
$C'=2GC\beta_t'/\sqrt{2Pa}$; $C''=2GC\beta_t''/\sqrt{2P}$.

In Fig. 14 values of C'' are plotted against the parameter (s/a). The same figure contains two more curves. One of them represents the distribution of the torque $M_t=C_1Pa$. The values of its ordinates are the sums of the values of the ordinates of curve M_{tP}/Pa, and of those of curve M_{tM}/M_{z0} of Fig. 5 multiplied by .175. The other curve represents the distribution of the unit angle of twist β_t'. It was calculated from the values of θ_M and θ_p shown in Figs. 9 and 10, respectively, with due consideration to Eq. 58. The ordinates are the values of C' of the equation

$$\beta_t'=C'\sqrt{2Pa/2GC} \quad . \quad . \quad . \quad . \quad . \quad (60)$$

With the aid of formulæ developed in Sec. A7 one gets the following values for the channel section used:—

$$\left.\begin{array}{l} e=.375 \text{ in.} \\ w=.625-s \text{ sq. in. for the upper flange} \\ w=.1875\,(2s-4) \text{ sq. in. for the back} \\ w=3.375-s \text{ sq. in. for the lower flange} \\ d=.25 \text{ in.} \end{array}\right\} \quad . \quad . \quad . \quad (61)$$

The maximum value of w is reached at the free edges. It is

$$w_{max} = \pm .625 \text{ sq. in.}$$

Thus the maximum direct stress due to non-uniform warping is given by the equation

$$\sigma_t = E w_{max} \beta_t'' = \pm \alpha P \text{ lb. per sq. in.}$$

where α is a function of the parameter s. Values of σ_t/P are plotted against (s/a) in Fig. 15.

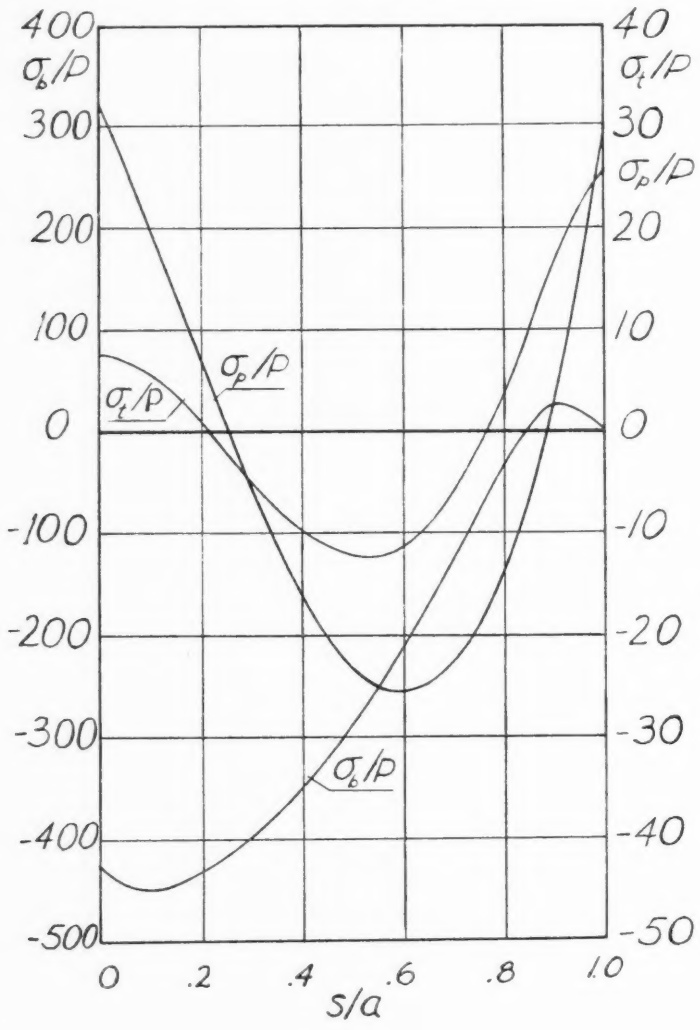

FIG. 15.

Numerical example of Sec. 10. Direct stress in edge fibre.

For the sake of comparison the direct stress caused by bending is also computed. The resultant bending moments can be obtained from the curves of Fig. 5 by adding up the values of the moments with the subscript P to the corresponding ones with the subscript M, the latter multiplied by .175 on account of Eq. 58. Dividing the moments so obtained by the section moduli

$$Z_p = .1067 \text{ in.}^3 \text{ and } Z_b = .0221 \text{ in.}^3$$

one obtains the direct stresses σ_p and σ_b, due to the bending about the two principal axes of inertia. These are also shown in Fig. 15.

Each of the three curves σ_t/P, σ_p/P, and σ_b/P represents the variation of the direct stress along one free edge of the channel caused by torsion with non-uniform warping, by bending about the principal normal, and by bending about the binormal of the median line of the bar, respectively, assuming the directions of the principal normal and of the binormal to be the directions of the two principal axes of inertia. It may be seen that the direct stresses caused by torsion and by bending about the principal normal are approximately of the same magnitude while those caused by bending about the binormal are about ten times higher. If one would turn the channel through a right angle so that the greater moment of inertia would correspond to the binormal, and the smaller to the principal normal, the stresses would naturally be reduced. Such a change in the orientation of the channel would couse only a slight change in the value of f_M, and Eq. 58 would become

$$M_{zo} = .185\, Pa$$

Thus the values of the bending moments, and of β_t'', would not change materially. Since, however, the ratio of the section moduli is 4.82, the stresses σ_p caused by bending about the principal normal would increase roughly to five fold, those caused by bending about the binormal would decrease roughly to one-fifth their original values. As σ_t would remain unchanged, its maximum value would be roughly one-fifth of the maximum bending stress. This is certainly high enough not to disregard except in rough approximate calculations.

The shearing stress may be calculated from Eq. A24 of the Appendix. Writing β for ϕ, one has

$$\tau = -E\beta_t''' \int_0^s w\, ds$$

The integral can be calculated if one substitutes the value of w from Eqs. 61 :—

$$\left.\begin{aligned} \int_0^s w\, ds &= .625\, s - .5\, s^2 \text{ for upper flange} \\ \int_0^s w\, ds &= .1875\, s^2 - .75\, s + .6875 \text{ for back} \\ \int_0^s w\, ds &= 3.375\, s - .5\, s^2 - 5.5 \text{ for lower flange} \end{aligned}\right\} \quad . \quad . \quad (62)$$

The value of this integral is proportional to the shearing stress, the distribution of which is shown in Fig. 16. The figure also contains the direct stress distribution caused by non-uniform warping.

The maximum value of the integral is .1953 in.^{3} at a distance of .625 in. from the free edge. Hence the value of the maximum shearing stress is

$$\tau_{max} = -.1953\, E\beta_t''' \quad . \quad . \quad . \quad . \quad . \quad (63)$$

It is possible to calculate β_t''' by differentiating, analytically or graphically, the function $\beta_t'' = \beta_t''(s)$ previously obtained and shown in Fig. 14. It is more accurate, however, to calculate it from Eq. 31a. Solving this for β_t''', one obtains

$$\beta_t''' = K^2\beta_t' - (1/E\Gamma)\, M_t(s)$$

Since the values of both β_t' and $M_t(s)$ were previously computed for a number of values of the parameter s and plotted in Fig. 14, β_t''' is easily obtained for the same values of s. The maximum shearing stress can then be calculated from Eq. 63.

At the same time shearing stresses are caused by the part of the torque that is resisted by simple (Saint-Venant) torsion. It is known from the theory of elasticity that the shearing stresses in a thin-walled channel section are very nearly the same as those in a narrow rectangular section the height of which is equal to the developed length of the median line of the channel, and the width of which is equal to the thickness of the channel. The maximum shearing stress occurs in the middle of the longer side of the rectangle. The value of the shearing stress due to simple torsion at $s=.625$ in. and $n=t/2$ does not differ much from

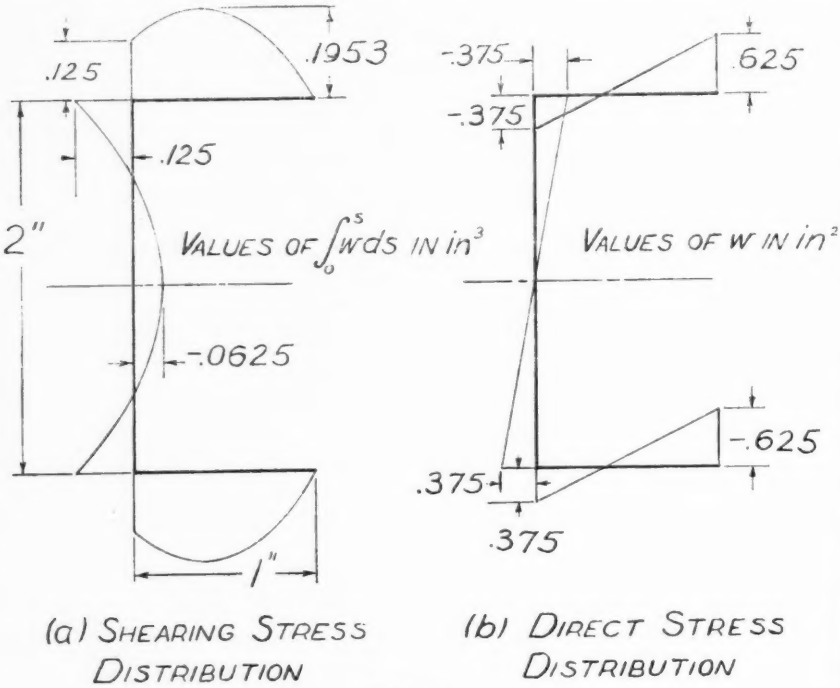

FIG. 16.

Stress distribution in a channel section due to non-uniform warping.

this maximum, as may be concluded from the statements made in Sec. A2 and from Fig. A3. Assuming that it is the same, and calculating the maximum stress for a rectangular section of a height $h=2$ ins. and a width $t=.04$ in., one may write

$$\tau=\beta_t' tG=6630\,C'P \text{ lbs. per sq. in.} \quad . \quad . \quad . \quad (64)$$

where C' is the non-dimensional coefficient plotted in Fig. 14.

The variation of the shearing stress in the most stressed fibre (in shear) is shown in Fig. 17. The total shearing stress is the sum of the two components. It is seen that in the case under consideration the maximum shear associated with non-uniform warping is roughly equal to that associated with simple (Saint-Venant) torsion. The maximum absolute value of the sum of the two is about

37 P lbs. per sq. in. If one calculates the maximum reduced stress for failure according to the Mohr-Guest theory,

$$\sigma_{red} = (\sigma^2 + 4\tau^2)^{\frac{1}{2}}$$

one finds that the shearing stress has a considerable influence upon failure.

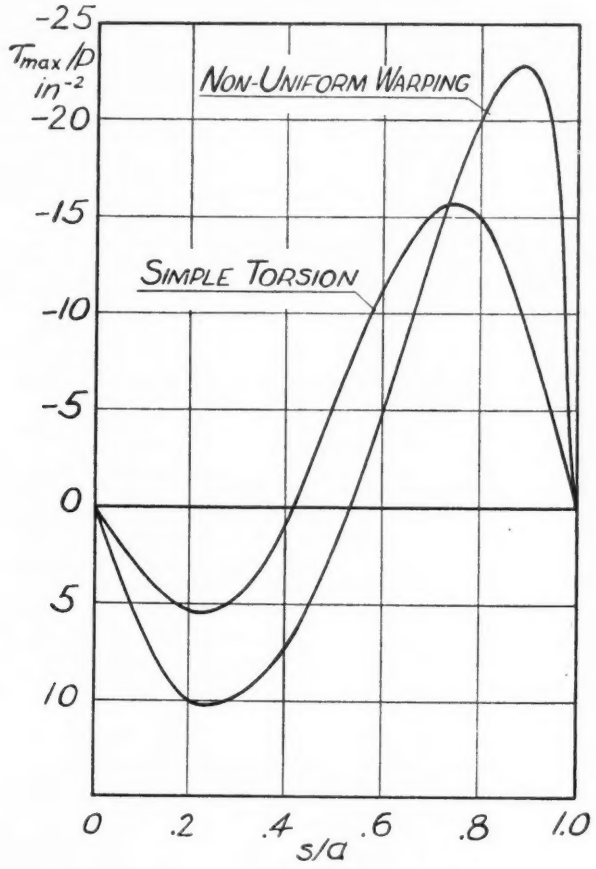

FIG. 17.

Numerical example of Sec. 10. Maximum shearing stress.

For a check, the torque caused by the two different kinds of shearing stress is calculated for $(s/a) = .8$. The shearing stress in simple torsion is $-15\,P$ lbs. per sq. in. The corresponding torque is

$$M_t = (1/3)\,Lt^2\tau_{max} = 2.133 \times 10^{-3}\tau_{max} = -.032\,P \text{ in. lb.}$$

The maximum shearing stress associated with non-uniform warping is $-20.4\,P$ lbs. per sq. in. Since the average shearing stress in the flange is .75 times the maximum, as may be calculated from the first of Eqs. 62, the total shear force S_F in the flange is

$$S_F = -.75 \times 20.4 \times 1 \times .04 = .612\,P$$

An equal and opposite shear force is acting in the lower flange; the moment of the two

$$M_t = -1.224\,P \text{ in. lb.}$$

The total moment is the sum of the two

$$M_{t\cdot tot} = -1.256\,P \text{ in. lb.}$$

Here the moment due to simple torsion is about 2.5 per cent. of the total; nowhere does it exceed 7 per cent. in this example. Nevertheless, as is seen from Fig. 17, the stresses associated with non-uniform warping are about equal to those due to simple torsion.

The torque at $(s/a)=.8$ is, from Fig. 14,

$$M_t = -.0623\,Pa = -1.246\,P$$

The deviation from the formerly calculated value is less than 1 per cent. This is regarded as satisfactory, since the calculations were carried out with the slide-rule.

It should be mentioned that in the example the numerical values obtained for the shearing stress associated with non-uniform warping are so high that the assumptions underlying the theory presented in the Appendix cannot be considered as satisfied. Thus the values of the stresses calculated for the channel section chosen are not reliable. Fortunately the sections actually used in aircraft construction are generally heavier than the section in the example; in them the shearing stresses associated with non-uniform warping are smaller, and consequently the theory of the Appendix applies.

11. Simplified Procedure.

A. If the shape of the centre line of the curved bar is reasonably similar to the intersection of two straight circular cylinders, then Marguerre's diagrams may be used for the determination of the unknown moment M_{z0}, of the moment distribution, and of the maximum moment, provided that the orientation of the principal axes of inertia of the section is such as assumed by Marguerre, or if not that it has little effect upon the moment distribution, and provided that the effect of non-uniform warping upon the unknown moment M_{z0} is negligibly small.

a. The directions of the principal axes of inertia permit the use of Marguerre's diagrams:—

1. If one of the principal axes of inertia is always horizontal, as assumed by Marguerre. This is the case, of course, if the section is circular, or circular cylindrical, since then every axis is a principal axis;

2. Or if the ratios $g=GC/EI_{p_1}$ and $h=GC/EI_{p_2}$ (where the numerator is the Saint-Venant torsional rigidity, and the denominators are the bending rigidities about the two principal axes of inertia) are both very small, say around or below .01.

For the solid circular, and the circular ring section these ratios are .77. For a solid, narrow rectangular section of a width b and a height h one has

$$C=(1/3)\,b^3h \quad I_{p_1}=(1/12)\,bh^3 \quad I_{p_2}=(1/12)\,b^3h$$
$$g=.385\times 4\,(b/h)^2 \quad h=.385\times 4=1.54$$

In the case of common thin-walled open sections g and h are usually small. For instance, in the case of the angle section shown in Fig. A5 the moment of inertia about the horizontal principal axis is

$$I=(1/12)\,h^3t\,[1+6\,(b/h)]$$

The torsional constant C is given by the formula

$$C=(1/3)\,ht^3\,[1+2\,(b/h)]$$

Thus

$$g = 1.54\,(t/h)^2\,[1 + 2\,(b/h)]/[1 + 6\,(b/h)]$$

For the heavy channel 1 in. by $\frac{1}{2}$ in. by $\frac{1}{8}$ in. g becomes .012; for the light channel 2 ins. by 1 in. by $\frac{1}{16}$ in. the value of g is .00075. Although the value of h is greater, its order of magnitude is the same.

b. Marguerre's diagrams should not be used because of the effects of non-uniform warping if the resistance of the section to non-uniform warping is considerably greater than its resistance to Saint-Venant torsion, and:—

1. If there is reason to believe that non-uniform warping influences the distortions caused by the external moments differently from the way it influences the distortions caused by the unknown moment (see Sec 9);

2. Or if g and h are not very small.

It may be seen from Secs. 7 and 8 that the effect of non-uniform warping upon the rigidity of a bar depends upon the value of the non-dimensional coefficient Ka. The probable range of this coefficient is, therefore, now established. In the case of the channel section of Fig. A5 the warping constant Γ is given in Eqs. A47 and A47*a*. Hence

$$(GC/E\Gamma) = 1.54\,(t^2/h^4)\,(h/b)^3\,[1 + 2\,(b/h)]\,[1 + 6\,(b/h)]/[2 + 3\,(b/h)]$$

The value of $K = (GC/E\Gamma)^{\frac{1}{2}}$ is, therefore, .66 for the above-mentioned "heavy" section, and .083 for the "light" section. Taking the length a of one-quarter of the centre line of the bar 15 ins. in the former case, and 12 ins. in the latter, one obtains the approximate limits of Ka:—

$$1 \leq Ka \leq 10$$

In the case of the angle section shown in Fig. A6*b* one may calculate the value of K with the aid of Eq. A70:—

$$K = 2.15\,(a + b)^{\frac{1}{2}}/(a^3 + b^3)^{\frac{1}{2}}$$

Thus for most angle sections used in aircraft K is greater than unity, and for most space curved rings Ka is greater than ten.

Numerical calculations show that the increase in stiffness of a twisted bar due to its resistance to non-uniform warping is negligibly small under normal loading conditions if the value of Ka is above 10; as was found in Sec. 8, it is very great if Ka is around 1. In the range from one to ten the effect of non-uniform warping upon the stiffness changes very rapidly.

B. The procedure developed in Secs. 3 and/or 4 should be used for determining M_{z0} and the moment distribution, if Marguerre's diagram cannot be used because of the shape of the centre line of the bar, or because of the reasons enumerated under *A*, and if it can be assumed that the effect of non-uniform warping on M_{z0} is small.

C. The procedure developed in Secs. 7 and 8 should be used if the effect of non-uniform warping upon M_{z0} is great. This is the case when the conditions stated under *B* hold.

D. For an accurate determination of both the direct and shearing stresses the distribution of the angle of twist must be calculated from the differential equation given in Sec. 7. This done, the stresses may be calculated as shown in Sec. 10. For an approximate calculation, the effects of non-uniform warping may be neglected if Ka is great; if it is very small, and thus the shearing stresses associated with non-uniform warping are very large, the theory developed in the Appendix cannot be relied on.

E. In the calculation of the distortions, as a rule, the effects of non-uniform warping must not be neglected if a thin-walled open section is used.

APPENDIX.

A1. SOLUTIONS OF THE PROBLEM OF TORSION.

Since de Saint-Venant established the basis of the present day theory of the torsion of straight bars, the shearing stress distribution in and the angle of twist of twisted bars has been calculated for a great number of different cross sections. Some important assumptions underlying these solutions are that the twisting moment is constant along the bar, it is applied by means of two equal and opposite couples at the two end sections of the bar, that these external couples are the resultants of shear stresses distributed over the end sections in the same way as are the shear stresses in any and every cross section of the twisted bar, and that the end sections are free to warp. This last assumption means that the points in the plane of the end sections are not constrained to remain in the plane when the bar is twisted. In this Appendix stresses, angle of twist, etc., calculated in such a way will be marked " in torsion according to Saint-Venant," or " according to Saint-Venant."

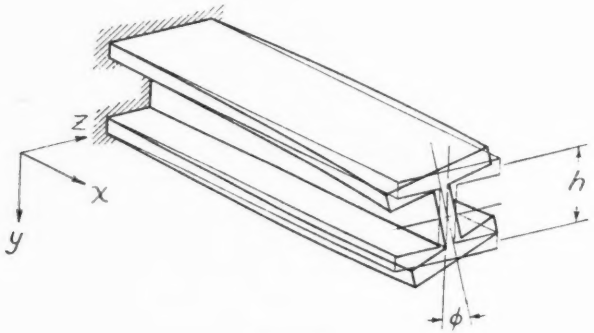

FIG. A1.

Torsion with non-uniform warping.

If these assumptions do not apply, the stress distribution in and the torsional rigidity of the twisted bar may be greatly affected. A problem of this kind was first discussed in 1905 in Russian, and in 1910 in German in papers written by S. Timoshenko,* in which he calculated the effect upon a twisted I-beam of preventing its fixed end section from warping. Fig. A1 shows that in such a case the distortions of the twisted bar involve a so-called differential bending, namely, the bending of the two flanges in opposite directions, in addition to the twisting according to Saint-Venant. Consequently direct stresses as well as shearing stresses are caused by the torque. This more complex problem is governed by the differential equation

$$T = GC\phi' - (h^2/2)\, EI_F\phi''' \quad . \quad . \quad . \quad . \quad . \quad (A1)$$

where T is the torque, GC the torsional rigidity of the bar according to Saint-Venant, E Young's modulus, I_F the moment of inertia of the cross section of one flange about an axis through its centre of gravity parallel to the y-direction, ϕ the angle of twist, and the dash denotes a differentiation with respect to x.

Eq. A1 was developed in Timoshenko's above-mentioned papers. A derivation and discussion of this equation will not be given here, since they can be found in Ref. 5.

* See Ref. 4. Also p. 282 of Ref. 5.

A2. Stresses in a Bar of Thin-Walled Open Section Twisted about its Centroidal Axis.

It is not difficult to establish an equation similar to Eq. A1 for a twisted bar which has a thin-walled open section of the kind used in aircraft construction. Two papers written in German by H. Wagner and R. Kappus, respectively, and published by the N.A.C.A. in English translation, deal with this subject. (See Refs. 7 and 2.) Notwithstanding the fact that the English translations are easily available, a derivation of the differential equation of torsion with non-uniform warping of thin-walled open sections will be given here since the publications are not easy to follow. One cause of this lack of clarity is that reference is made to a paper by W. Lücker for some of the proofs, and this paper is unknown to the writer and not available in this country.

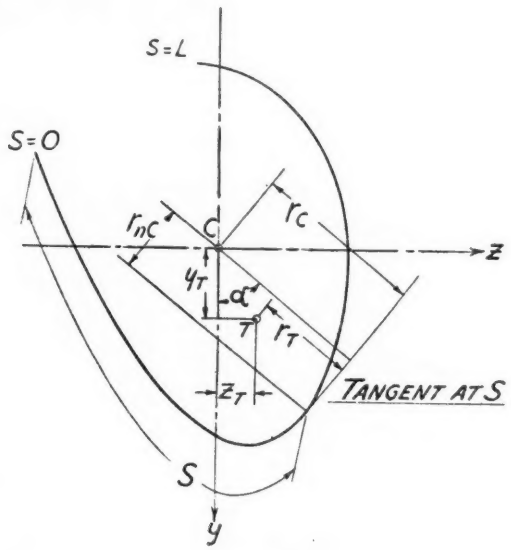

Fig. A2.

Median line of a thin-walled open section.

Fig. A2 represents the median line of the cross section of a thin-walled open section bar. The x-axis is coincident with the longitudinal axis of the bar and points toward the reader so that the axes x, y, z represent a right-hand system. The position of a point of the cross section is essentially determined by its distance s from the point $s=0$, measured along the median line, since the wall thickness t is considered small. The positive direction of the tangent is determined by the rule that the tangent should point in the direction of increasing values of the co-ordinate s. The perpendicular distance r_c of the centroid from the tangent is taken positive if the moment of the tangent vector about the centroid is as shown in the figure, that is when it corresponds to a rotation from the y-axis toward the z-axis. Furthermore, the bar, the cross section of which is shown in Fig. A2, is assumed to be straight, of constant cross section, and made of homogeneous isotropic material which follows Hooke's law.

Let it be assumed that two cross sections of the bar a distance dx apart rotate about the x-axis, which passes through the centroid, by an angle $d\phi$ relative to

each other. Such a rotation will cause shearing stresses to appear in the cross sections which may preferably be calculated with the aid of the membrane analogy.* Fig. A3 shows the contour lines of a membrane, attached to the edges of an open section which has a wall thickness t, as they develop under uniform lateral pressure. These contour lines can also be considered the lines of shearing stress of the twisted bar. It may be seen that no shearing stresses are acting along the median line of the thin wall. From this, however, it follows that the originally right angle between any line element ds of the median line and the direction of the longitudinal fibres of the bar, originally going in the x-direction, must remain a right angle after distortions, since by the laws of the theory of elasticity the distortion of this angle is proportional to the shearing stress which acts in the cross section in the direction of the median line. This requirement permits the calculation of the warping of the cross section.

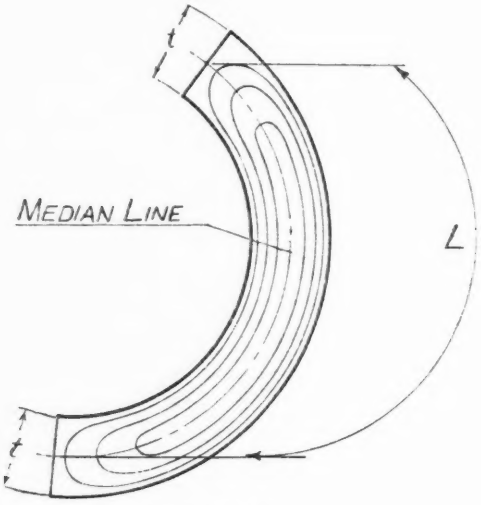

FIG. A3.
Lines of shearing stress in a twisted thin-walled open section.

Let u denote the displacement of a point of the median line of the cross section from its original position in the x-direction, and v that in a direction tangential to the median line. Then the change γ of the originally right angle between the longitudinal fibres of the bar and the tangent to the median line of the cross section is given by the known equation of the theory of elasticity

$$\gamma = (\partial v/\partial x) + (\partial u/\partial s) \quad . \quad . \quad . \quad . \quad (A.2)$$

This change, however, must vanish as stated above and therefore u can be obtained by an integration

$$u = -\int (\partial v/\partial x)\, ds + F(x) \quad . \quad . \quad . \quad . \quad (A.3)$$

if $\partial v/\partial x$ is known. In Eq. A.3 $F(x)$ is an arbitrary function of x. Since v is the displacement of a point of the median line in the direction of the tangent to the median line, and since the relative displacement of the two neighbouring sections, by assumption, consists of a rotation about the centroid of the section,

$$dv = r_c d\phi = \phi' r_c dx \quad . \quad . \quad . \quad . \quad (A.4)$$

* See p. 239 of Ref. 6.

where the dash again denotes a differentiation with respect to x, and r_c is a function of the co-ordinate s alone. When dv is introduced from Eq. A.4 into Eq. A.3 the longitudinal displacement u becomes

$$u = -\phi' \int_0^s r_c ds + F(x) \quad . \quad . \quad . \quad . \quad . \quad (A.5)$$

where $F(x)$ is a function of x alone.

In further calculations it will be convenient to measure the longitudinal displacements from an average-displacement-plane rather than from the original position of the points, and to define a quantity w_c as such a displacement corresponding to a unit rate of change of the angle of twist :—

$$w_c = -\int_0^s r_c ds + C \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.6)$$

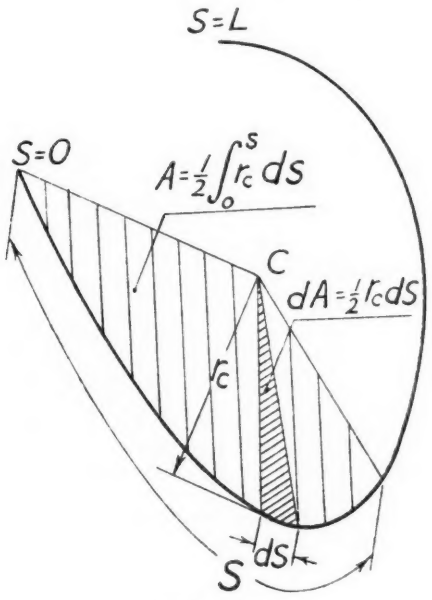

FIG. A4.

Graphical determination of the warping (*Eqs.* A.6).

where

$$C = (1/L) \int_0^L ds \int_0^s r_c ds \ *$$

Here the warping w_c is a function of s alone and is determined by the shape of the median line of the cross section. It can be readily obtained by a graphical integration, since $\int r_c ds$ is the double area enclosed by the segment of the median

* Throughout this paper this form of setting the double integral is the same as $\int_0^L \int_0^s r_c ds ds$

line between o and s, and by the two straight lines connecting these two end points to the centroid of the section. (See Fig. A4.)

The strain in the x-direction is, by definition,

$$\epsilon_x = (\partial u / \partial x)$$

and the longitudinal stress

$$\sigma_x = E\,(\partial u/\partial x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.7)$$

Eqs. A.5, A.6, and A.7 yield

$$\sigma_x = \phi'' w_C E + f(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.8)$$

where $f(x)$ is an arbitrary function of x replacing the more complex expression $E\,[F'(x) - C\phi'']$. It can be determined from the requirement that for the case of pure torsion the resultant axial force must vanish. If one considers that the stress must be substantially constant across the small constant wall thickness t, the resultant axial force X becomes

$$X = t\int_0^L \sigma_x ds = \phi'' tE \int_0^L w_C ds + tLf(x)$$

Since from Eq. A.6 it follows that

$$\int_0^L w_C ds = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.9)$$

the condition of a vanishing axial force X is

$$f(x) = 0.$$

Hence the axial stress distribution is defined by the equation

$$\sigma_x = \phi'' w_C E \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.10)$$

where w_C is the quantity defined in Eq. A.6.

A3. The Centre of Twist.

The stress distribution calculated above corresponds to a vanishing resultant axial force X, but in general the resultant moment of such stresses does not vanish about either the y- or the z-axis. The equilibrium conditions cannot be satisfied, therefore, if the external force system consists of a pure torque. This contradiction may be eliminated by assuming that the bar does not twist about an axis through the centroid but rather about an axis through a point T. The co-ordinates y_T, z_T of this point T can be determined from the requirements that the resultant moment should vanish about both the y- and z-axis.

For a rotation about an axis through T the distance r_C from the centroid must be replaced by the distance r_T from point T in the formulas developed in Sec. A2 (see Fig. A2):—

$$r_T = r_C - y_T \cos\alpha - z_T \sin\alpha \quad . \quad . \quad . \quad . \quad . \quad (A.11)$$

and a warping w_T may be defined in analogy to w_C:—

$$\left.\begin{aligned} w_T &= -\int_0^s r_T ds + (1/L)\int_0^L ds \int_0^s r_T ds \\ &= -\int_0^s (r_C - y_T \cos\alpha - z_T \sin\alpha)\, ds \\ &\quad + (1/L)\int_0^L ds \int_0^s (r_C - y_T \cos\alpha - z_T \sin\alpha)\, ds \end{aligned}\right\} \quad . \quad . \quad (A.12)$$

With this definition obviously $\int_0^L w_T ds = 0. \quad . \quad . \quad . \quad . \quad . \quad (A.13)$

Considerations similar to those made under Sec. A2 lead to the result that the axial stress σ_x is

$$\sigma_x = \phi'' w_T E, \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.14)$$

and that with such a stress distribution the resultant axial force vanishes.

With the aid of the identities

$$ds \cos \alpha = dz \text{ and } ds \sin \alpha = -dy \quad . \quad . \quad . \quad (A.15)$$

Eq. A.12 can be transformed:—

$$w_T = -\int_0^s r_c ds + y_T \int_0^z dz - z_T \int_0^y dy$$

$$+ (1/L) \int_0^L \left[\int_0^s r_c ds - y_T \int_0^z dz + z_T \int_0^y dy \right] ds$$

$$= w_c + y_T (z - z_o) - z_T (y - y_o) - (1/L) \int_0^L [y_T (z - z_o) - z_T (y - y_o)] \, ds$$

since the sum of the original first and fourth terms is w_c on account of Eqs. A.6. This equation can be further simplified by evaluating the definite integrals:—

$$(1/L) \int_0^L y_T z ds = (1/L) \int_0^L z_T y ds = 0$$

$$(1/L) \int_0^L y_T z_o ds = y_T z_o$$

$$(1/L) \int_0^L z_T y_o ds = z_T y_o$$

This is true since z and y are measured from centroidal axes and since the symbols with a subscript are constants. With these values the warping w_T caused by a rotation about a longitudinal axis through point T becomes:—

$$w_T = w_C + y_T z - z_T y, \quad . \quad . \quad . \quad . \quad . \quad (A.16)$$

which equation means that the warping w_T differs from the warping w_c by an amount corresponding to a rotation of the plane of the cross section an axis contained in the plane. Upon substitution into Eq. A.14 of w_T from Eq. A.16 the stress becomes:—

$$\sigma_x = \phi'' E \, [w_c + y_T z - z_T y] \quad . \quad . \quad . \quad . \quad (A.17)$$

The unknown co-ordinates y_T and z_T can be calculated from the requirement that the resultant moment must vanish about both the y- and z-axis:—

$$t \int_0^L \sigma_x y ds = 0 \text{ and } t \int_0^L \sigma_x z ds = 0 \quad . \quad . \quad . \quad (A.18)$$

If from Eq. A.14 σ_x is introduced into these equations they become:—

$$E \phi'' t \int_0^L w_T y ds = 0 \text{ and } E \phi'' t \int_0^L w_T z ds = 0 \quad . \quad . \quad . \quad (A.18a)$$

where w_T is the warping given by Eq. A.16. With the symbols for the second moments of the cross section

$$t \int_0^L y^2 ds = I_{yy} \quad t \int_0^L z^2 ds = I_{zz} \quad t \int_0^L yz ds = I_{yz} \quad . \quad . \quad . \quad (A.19)$$

and for the " warping moments "

$$F_y = t\int_0^L w_C y\,ds \text{ and } F_z = t\int_0^L w_C z\,ds \quad . \quad . \quad . \quad (A.20)$$

Eqs. A.18 become:—

$$\left.\begin{aligned} y_T I_{yz} - z_T I_{yy} &= -F_y \\ y_T I_{zz} - z_T I_{yz} &= -F_z \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (A.21)$$

Eqs. A.21 can be solved for y_T and z_T:—

$$\left.\begin{aligned} y_T &= -(F_z I_{yy} - F_y I_{yz})/(I_{yy} I_{zz} - I_{yz}^2) \\ z_T &= -(F_y I_{zz} - F_z I_{yz})/(I_{yy} I_{zz} - I_{yz}^2) \end{aligned}\right\} \quad . \quad . \quad . \quad (A.22)$$

The point defined by the co-ordinates y_T, z_T given by Eqs. A.22 is called the centre of twist. The cross section of the twisted bar must rotate about an axis passing through the centre of twist if the external load is a pure torque and if the bar cannot warp uniformly so that in addition to the shearing stresses axial stresses are caused in the cross section. It should be noted that in Eqs. A.19-A.22 y and z are measured from the centroidal axes, and w_C must be calculated from Eqs. A.6 for a rotation about the centre of gravity of the section.

In the above developments it was tacitly assumed that the shearing stresses caused by non-uniform warping were small as compared to those due to Saint-Venant torsion. Only in this case is Fig. A3 representative of the shearing stress distribution in the twisted bar.

A4. Shear Stresses Due to Restrained Warping.

An equilibrium condition of an infinitesimal element of the thin wall is

$$(\partial\sigma_x/\partial x) + (\partial\tau/\partial s) = 0 \quad . \quad . \quad . \quad . \quad (A.23)$$

This differential equation permits the calculation of the shearing stresses in the thin wall, since the distribution of the axial stresses is already known. By introducing σ_x from Eq. A.14 into Eq. A.23, and solving the equation, the following expression for the shearing stress is obtained:—

$$\tau = -\int_0^s (\partial\sigma_x/\partial x)\,ds + g\,(x)$$

$$= -\phi''' E\int_0^s w_T ds + g\,(x)$$

where $g\,(x)$ is an arbitrary function of x, to be determined from the boundary conditions. Since no external shearing forces are introduced along the edges $s=0$ and $s=L$, the shearing stress must be zero at the edges:—

$$\tau = 0 \text{ for } s=0 \text{ and } s=L.$$

These conditions can be satisfied if

$$g\,(x) = 0,$$

since Eq. A.13 holds true. Consequently the shear distribution is given by

$$\tau = -E\phi'''\int_0^s w_T ds \quad . \quad . \quad . \quad . \quad . \quad (A.24)$$

It can easily be shown that the resultant of these shearing stresses is a pure torque. The y-component of the resultant is

$$S_y = -\int_A \tau \sin\varkappa\, dA = -t\int_0^L \tau \sin\varkappa\, ds = E\phi''' t\int_0^L \sin\varkappa\, ds \int_0^s w_T ds.$$

This integral can be evaluated by integrating by parts:—

$$S_y = E\phi'''t \left\{ \left[-y \int_0^s w_T ds \right]_0^L + \int_0^L w_T y ds \right\}.$$

In this equation the upper limit of the first term is zero in accordance with Eq. A.13, and the lower limit is equal to zero for obvious reasons. The second term vanishes in accordance with the first of Eqs. A.18*a*. Hence

$$S_y = 0 \qquad . \quad . \quad . \quad . \quad . \quad . \quad (A.25)$$

Similarly

$$S_z = \int_A \tau \cos \varkappa dA = t \int_0^L \tau \cos \varkappa ds = -E\phi'''t \int_0^L \cos \varkappa ds \int_0^s w_T ds$$

becomes

$$S_z = -E\phi'''t \left\{ \left[z \int_0^s w_T ds \right]_0^L - \int_0^L z w_T ds \right\}$$

which vanishes for reasons similar to those shown above, giving

$$S_z = 0 \qquad . \quad . \quad . \quad . \quad . \quad . \quad (A.26)$$

Having thus ascertained that there is no resultant shear force due to restrained warping, one can proceed to the calculation of the resultant torque:—

$$T = t \int_0^L \tau r_T ds = -E\phi'''t \int_0^L r_T ds \int_0^s w_T ds.$$

Integrating by parts, one obtains the integral

$$T = -E\phi'''t \left\{ \left[-w_T \int_0^s w_T ds \right]_0^L + \int_0^L w_T^2 ds \right\}$$

Since $\int_0^s w_T ds$ vanishes for the upper limit by reason of Eq. A.13, and for the lower limit for obvious reasons, the above equation can be rewritten in the form

$$T = -E\phi'''\Gamma \qquad . \quad . \quad . \quad . \quad . \quad . \quad (A.27)$$

where

$$\Gamma = t \int_0^L w_T^2 ds \qquad . \quad . \quad . \quad . \quad . \quad . \quad (A.28)$$

Γ will be called the warping constant.* From the definition of w_T it follows that the warping constant is a cross-sectional constant, it has the dimension of a length to the sixth power, and is always positive.

A5. The Differential Equation of Torsion with Non-Uniform Warping for Thin-Walled Open Sections.

The findings of the above sections show that in general if the torque acting upon a straight thin-walled open section bar varies along the bar, or if the free warping of any cross section is prevented, axial stresses will appear in the thin wall in addition to the shearing stresses according to Saint-Venant. These axial stresses will always be accompanied by shearing stresses, the distribution and magnitude of which are entirely different from those according to Saint-Venant. The shearing stresses associated with restrained warping add up to a pure torque which together with the resultant torque of the Saint-Venant shearing stresses balances the external torque applied to the bar.

* This symbol was used by Goodier in Ref. 1.

Since the torque due to the shearing stresses according to Saint-Venant is

$$T_{SV} = GC\phi' \qquad \text{(A.29)}$$

where GC is the torsional rigidity according to Saint-Venant, the connection of the external torque with the internal torque (due to stresses) can be written, with the aid of Eqs. A.27-A.29,

$$T = GC\phi' - E\Gamma\phi''' \qquad \text{(A.30)}$$

This is the differential equation of torsion with non-uniform warping for thin-walled open sections and is very similar to Eq. A.1. If the geometry and the material properties of the bar, and the variation of the external moment are known, the angle of twist can be calculated by solving this differential equation.

A6. The Shear Centre.

For numerical computations it is helpful to know that the centre of twist of a section is identical with its shear centre. This follows from the reciprocal theorem. It is interesting, however, to prove formally that the requirements for the shear centre of a thin-walled open section lead to the same formulæ as those found previously when the centre of twist was determined. The shear centre is defined in the theory of bending of non-symmetrical sections as the point through which an external shear force must pass if it is to cause bending alone and no torsion. Alternatively it can be stated that the resultant shear force always passes through the shear centre, if the section is only bent and not twisted. This second definition may be used advantageously to calculate the location of the shear centre.

If the direct stress is known, the shear stress distribution can be calculated by solving the differential equation

$$\partial\sigma_x/\partial x + \partial\tau/\partial s = 0 \qquad \text{(A.23)}$$

From this

$$\tau = -\int_0^s (\partial\sigma_x/\partial x)\, ds + G(x)$$

where $G(x)$ is an arbitrary function of x alone.

The axial stress in a non-symmetrical beam bent by two moments M_y and M_z is given by the formula

$$\sigma_x = z\,(I_{yy}M_y + I_{yz}M_z)/(I_{yy}I_{zz} - I_{yz}^2) - y\,(I_{zz}M_z + I_{yz}M_y)/(I_{yy}I_{zz} - I_{yz}^2)$$

where y and z are any two mutually perpendicular centroidal axes in the plane of the cross section, M_y and M_z the applied bending moments whose vectors are parallel to the y- and z-axes, respectively, and point in their positive directions (whereby the sense of the vectors corresponds to the right-hand screw rule), and the cross-sectional properties of the bar are defined by the integrals

$$I_{yy} = \int_A y^2 dA = t\int_0^L y^2 ds$$

$$I_{zz} = \int_A z^2 dA = t\int_0^L z^2 ds$$

$$I_{yz} = \int_A yz\, dA = t\int_0^L yz\, ds$$

Since

$$(dM_y/dx) = S_z \text{ and } (dM_z/dx) = -S_y \qquad \text{(A.31)}$$

where S_z is the resultant shear force acting parallel to the z-axis, and S_y the resultant shear force acting parallel to the y-axis, the differential coefficient of the axial stress is

$$(\partial\sigma_x/\partial x) = z\,(I_{yy}S_z - I_{yz}S_y)/(I_{yy}I_{zz} - I_{yz}) + y\,(I_{zz}S_y - I_{yz}S_z)/(I_{yy}I_{zz} - I_{yz}^2)$$

For a given bar and given loading every symbol but y and z of the above formula represents a constant. With the introduction of the symbols

$$\left.\begin{aligned} C_1 &= (I_{yy}S_z - I_{yz}S_y)/(I_{yy}I_{zz} - I_{yz}^2) \\ C_2 &= (I_{zz}S_y - I_{yz}S_z)/(I_{yy}I_{zz} - I_{yz}^2) \end{aligned}\right\} \quad . \quad . \quad . \quad (A.32)$$

the expression for the shearing stress becomes

$$\tau = -\int_0^s (C_1 z + C_2 y)\,ds + G\,(x) \quad . \quad . \quad . \quad . \quad (A.33)$$

The boundary conditions are

$$\tau = 0 \text{ for } s = 0 \text{ and } s = L.$$

They are satisfied if

$$G\,(x) = 0,$$

since

$$\int_0^L C_1 z\,ds = (1/t)\int_A C_1 z\,dA = 0$$

$$\int_0^L C_2 y\,ds = (1/t)\int_A C_2 y\,dA = 0$$

for the reason that y and z are measured from centroidal axes. Consequently Eq. A.33 reduces to

$$\tau = -\int_0^s (C_1 z + C_2 y)\,ds \quad . \quad . \quad . \quad . \quad (A.34)$$

wherein C_1 and C_2 are constants defined by Eqs. A.32.

Shearing stresses distributed over the cross section according to Eq. A.34 add up, in general, to a resultant shearing force having a y- and a z-component, and to a resultant torque. With a view to checking Eq. A.34 the y-component of the resultant shearing force will now be calculated.

$$\int_A \tau_y dA = -\int_A \tau \sin\alpha\,dA = -t\int_0^L \tau \sin\alpha\,ds$$

$$= t\int_0^L \sin\alpha\,ds \int_0^s (C_1 z + C_2 y)\,ds.$$

Integrating by parts, one obtains

$$\int_A \tau_y dA = t\,\{\,[y\tau]_0^L + \int_0^L y\,(C_1 z + C_2 y)\,ds\,\}.$$

On the right-hand side the first term vanishes since τ vanishes for both the upper and lower limit. The second term becomes

$$C_1 I_{yz} + C_2 I_{yy}.$$

With the substitution of C_1 and C_2 from Eqs. A.32 the integral becomes

$$\int_A \tau_y dA = S_y \quad . \quad . \quad . \quad . \quad . \quad (A.35)$$

In a similar way it can be shown that

$$\int_A \tau_z dA = S_z \qquad . \qquad . \qquad . \qquad . \qquad . \qquad (A.36)$$

Consequently the shearing stresses in the section, as given by Eq. A.34, actually add up to the total applied shearing force. Thus the only remaining task is to determine point S in the plane of the cross section about which the resultant torque T of all the shearing stresses vanishes. By definition this point is the shear centre of the cross section, since the moment of the shearing stresses vanishes identically only for the point through which the resultant shear force passes for any arbitrary choice of S_y and S_z. Denoting by r_s the perpendicular distance of the shear centre from the tangent to a point of the median line of the thin wall of the section, one obtains the equation of the moment about the shear centre :—

$$T = t\int_0^L \tau r_s ds = -t\int_0^L r_s ds \int_0^s (C_1 z + C_2 y)\, ds.$$

Integrating by parts, one obtains :—

$$T = -t\,\{\,[w_s \tau]_0^L + \int_0^L (C_1 z + C_2 y)\, w_s ds\,\}\,.$$

In this equation the first term on the right-hand side vanishes since τ vanishes for both limits. Since C_1 and C_2 are entirely arbitrary for the reason that S_y and S_z may have any arbitrary value, the torque T vanishes identically only if

$$t\int_0^L C_1 z w_s ds = t\int_0^L C_2 y w_s ds = 0 \qquad . \qquad . \qquad . \qquad . \qquad (A.37)$$

These two conditions determine the location of the shear centre. Eqs. A.37, however, identical with Eqs. A.18*a* except for a constant factor. Consequently the requirements for the shear centre and those for the centre of twist are identical, and therefore the centre of twist must coincide with the shear centre.

A7. Example for the Calculation of the Warping Constant.

The first step in calculating the warping constant Γ is to determine the centre of twist. This may be done with the aid of Eqs. A.22 but in most cases it is more convenient to start from the definition of the shear centre. This definition was given and applied to the general thin-walled section in Sec. A6. Practical calculations for commonly used sections were carried out by Timoshenko in Art. 8 of Ref. 5.

The second step is to determine the warping w_T for a rotation about the centre of twist. This can always be done easily for thin-walled sections of any arbitrary shape by the graphical method explained in Sec. A2 and shown in Fig. A4. When this is accomplished, a numerical or graphical integration of the integrand in Eq. A.28 will not present any difficulties. It should be noted, however, that the formulæ in Sec. A2, and Fig. A4 refer to a rotation about the centroid, and not about the centre of twist as required here.

In the case of sections of a shape which can be represented by simple mathematical expressions, the warping constant Γ can be found analytically. This will be shown in the example of the channel section of Fig. A5. On the assumption of a small wall thickness t the moment of inertia of the section is

$$I_{yy} = (t/12)\,(6bh^2 + h^3) \qquad . \qquad . \qquad . \qquad . \qquad (A.38)$$

The distance e of the centre of twist from the back of the channel is

$$e=(tb^2h^2/4I) \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.39)$$

according to the formula on p. 53 of Ref. 5. This can also be written

$$e=3b^2/(6b+h) \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.39a)$$

Since r_T remains constant along the two flanges and along the back, the first integral on the right-hand side of Eq. A.12 becomes:—

$$\left.\begin{aligned} -\int_0^s r_T ds &= -(h/2)\,s \text{ when } 0\leq s\leq b \\ -\int_0^s r_T ds &= es-(b/2)\,(2e+h) \text{ when } b\leq s\leq b+h \\ -\int_0^s r_T ds &= -(h/2)\,[s-(2e+h)] \text{ when } b+h\leq s\leq 2b+h \end{aligned}\right\} . \quad (A.40)$$

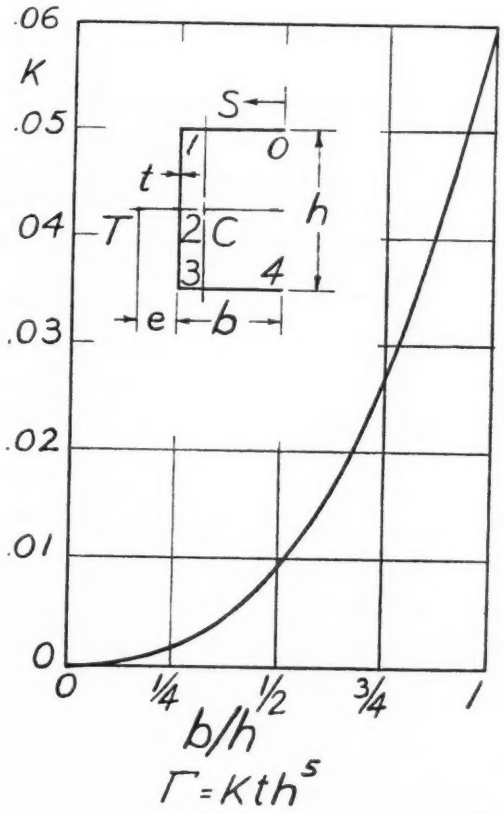

FIG. A5.

Warping constant of the channel section.

With these values the second integral on the right-hand side of Eq. A.12 becomes

$$(1/L)\int_0^L ds \int_0^s r_T ds = (h/2)(b-e) \quad . \quad . \quad . \quad . \quad (A.41)$$

The warping w_T for a rotation about the centre of twist, therefore, is given by the expressions

$$\left.\begin{aligned} w_T &= (h/2)(D-s) \text{ when } 0 \leq s \leq b \\ w_T &= (e/2)(2s-L) \text{ when } b \leq s \leq b+h \\ w_T &= (h/2)(L-D-s) \text{ when } b+h \leq s \leq 2b+h \end{aligned}\right\} \quad . \quad . \quad (A.42)$$

where L is the total length of the centre line of the section

$$L = 2b+h \quad . \quad . \quad . \quad . \quad . \quad (A.43)$$

and D denotes the expression

$$D = b-e \quad . \quad . \quad . \quad . \quad . \quad (A.43a)$$

The calculation of Γ according to Eq. A.28 is now straightforward. By symmetry

$$\int_0^L w_T^2 ds = 2\int_0^b w_T^2 ds + \int_0^{b+h} w_T^2 ds \quad . \quad . \quad . \quad (A.44)$$

The integration gives

$$2t\int_0^b w_T^2 ds = (h^2 t/6)[3D^2 b - 3Db^2 + b^3]$$

$$t\int_b^{b+h} w_T^2 ds = (t/12)e^2 h^3$$

After some algebraic transformations the warping constant becomes

$$\Gamma = (th^2/12)[e^2 h + 2b^3 - 6eb(b-e)] \quad . \quad . \quad . \quad (A.45)$$

If the value of e is introduced into this formula from Eq. A.39*a*, the warping constant can be written as

$$\Gamma = (th^2 b^3/24)[1 + 3h/(6b+h)] \quad . \quad . \quad . \quad (A.46)$$

If this expression for Γ is introduced into Eq. A.27, it can be easily verified that the result is identical with Eq. (*y*) on p. 292 of Ref. 5. A more convenient form for numerical computations is

$$\Gamma = kth^5 \quad . \quad . \quad . \quad . \quad . \quad (A.47)$$

where

$$k = (1/12)[3(b/h)^4 + 2(b/h)^3]/[6(b/h)+1] \quad . \quad . \quad (A.47a)$$

The value of k is plotted in the chart of Fig. A.5.

A8. The Effect of a Non-Vanishing Wall Thickness.

A refinement of the foregoing calculations may be accomplished by a consideration of the effect of a non-vanishing wall thickness. Wagner showed this in Ref. 7. This effect is of importance with sections for which Γ (Eq. A.28) vanishes, as is the case for instance with angle- and T-sections.

The distance of a point of the section from the median line, measured perpendicularly to the tangent to the median line, is denoted n. The positive direction of the perpendicular may be obtained by rotating a vector which points in the positive direction of the tangent clockwise through a right angle; it is, therefore, the " outward " direction in Fig. A2. The distance of the perpendicular from the centroid is denoted r_{nC}; it is considered positive if a force pointing in the positive direction of the perpendicular causes a counter-clockwise rotation about the centroid. Consequently r_{nC} is positive in the case of point s shown

in Fig. A2. The displacement of a point from its original position during the elastic distortions measured in the positive direction of the perpendicular is denoted by the letter p.

From the basic equations of the theory of elasticity the change γ of the originally right angle between the perpendicular to the median line and the longitudinal fibres of the twisted bar is

$$\gamma=(\partial p/\partial x)+(\partial u/\partial n) \quad . \quad . \quad . \quad . \quad (A.48)$$

On the other hand it may be seen from Fig. A3 that except for a very limited region at the ends of the thin-walled section the direction of the Saint-Venant shearing stresses is parallel to the median line. If one assumes that torsion with non-uniform warping of a section with a non-vanishing wall thickness does not cause shearing stresses of any appreciable magnitude in the direction of the perpendicular, then the angle $\gamma=\tau/G$ must vanish. Hence

$$u=-\int_0^n(\partial p/\partial x)\,dn+u_0 \quad . \quad . \quad . \quad . \quad (A.49)$$

where u_0 is the longitudinal displacement of a point of the median line. From the assumptions concerning the distortions of the bar, stated in Sec. A2, it follows that

$$dp=r_{nC}d\phi=\phi' r_{nC}dx \quad . \quad . \quad . \quad . \quad (A.50)$$

Substituting into Eq. A.49, one obtains

$$u=-\phi'\int_0^n r_{nC}dn+u_0=-\phi' r_{nC}n+u_0 \quad . \quad . \quad . \quad (A.51)$$

since r_{nC} is a constant when one integrates with respect to u.

The displacement u_0 of the median line was calculated in Sec. 2. It is given in Eq. A.5 in which it is represented by the symbol u without the subscript o. Substituting its value into Eq. A.51, and using an asterisk to denote that the displacement u (or any other quantity in future calculations) was determined with due consideration of the effect of a non-vanishing wall thickness, one obtains

$$u^*=-\phi'\int_0^s r_C ds-\phi' r_{nC}n+F(x) \quad . \quad . \quad . \quad (A.52)$$

Since n is measured from the median line, it follows from Eq. A.51 that the average displacement across the wall thickness is the displacement at the median line. Consequently the warping w_C^* differs from the warping w_C defined in Sec. 2 only by a quantity corresponding to the first term on the right-hand side of Eq. A.51 :—

$$w_C^*=w_C-r_{nC}n \quad . \quad . \quad . \quad . \quad . \quad (A.53)$$

The warping w_C is given in Eqs. A.6. The direct stress σ_x^* can be calculated in the same way as in Sec. 2, and since the average stress across the wall is the stress at the median line, one obtains an equation analogous to Eq. A.10 :—

$$\sigma_x^*=\phi'' w_C^* E \quad . \quad . \quad . \quad . \quad . \quad (A.54)$$

If the cross sections of the bar rotate about the centre of twist T rather than about the centroid C, one may write

$$w_T^*=w_T-r_{nT}n \quad . \quad . \quad . \quad . \quad . \quad (A.55a)$$

and

$$\sigma_x^*=\phi'' w_T^* E \quad . \quad . \quad . \quad . \quad . \quad (A.55b)$$

The connection between the stresses corresponding to a rotation about the centre of twist and about the centroid can be established with the aid of Eq. A.16. Eq. A.55a may be written in the form

$$w_T^*=w_C+y_T z-z_T y-r_{nT}n$$

where y and z are co-ordinates of the median line measured from the centroid. Since the y- and z-components of n are

$$n_y = n \cos \alpha \quad \text{and} \quad n_z = n \sin \alpha$$

and further since

$$r_{nT} = r_{nC} - y_T \sin \alpha + z_T \cos \alpha \quad . \quad . \quad . \quad . \quad (A.56)$$

one obtains, with consideration of Eq. A.53,

$$w_T^* = w_C^* + y_T (z + n_z) - z_T (y + n_y) \quad . \quad . \quad . \quad (A.57)$$

In this equation $(y + n_y)$ and $(z + n_z)$ are the co-ordinates of any point of the cross section.

Moment equilibrium requires that

$$\int_0^L ds \int_{-t/2}^{t/2} \sigma_x^* (y + n_y)\, dn = 0 \quad . \quad . \quad . \quad . \quad (A.58)$$

and

$$\int_0^L ds \int_{-t/2}^{t/2} \sigma_x^* (z + n_z)\, dn = 0 \quad . \quad . \quad . \quad . \quad (A.58)$$

where σ_x^* has the value given in Eq. A.55b. These requirements permit the calculation of the co-ordinates y_T^* and z_T^* of the centre of twist:—

$$\left.\begin{aligned} y_T^* &= -(F_z^* I_{yy}^* - F_y^* I_{yz}^*)/(I_{yy}^* I_{zz}^* - I_{yz}^{2*}) \\ z_T^* &= (F_y^* I_{zz}^* - F_z^* I_{yz}^*)/(I_{yy}^* I_{zz}^* - I_{yz}^{2*}) \end{aligned}\right\} \quad . \quad . \quad (A.59)$$

where

$$\left.\begin{aligned} I_{yy}^* &= \int_0^L ds \int_{-t/2}^{t/2} (y + n_y)^2\, dn \\ I_{zz}^* &= \int_0^L ds \int_{-t/2}^{t/2} (z + n_z)^2\, dn \\ I_{yz}^* &= \int_0^L ds \int_{-t/2}^{t/2} (y + n_y)(z + n_z)\, dn \\ F_y^* &= \int_0^L ds \int_{-t/2}^{t/2} w_C^* (y + n_y)\, dn \\ F_z^* &= \int_0^L ds \int_{-t/2}^{t/2} w_C^* (z + n_z)\, dn \end{aligned}\right\} \quad . \quad . \quad . \quad (A.60)$$

The expressions given for the warping moments F_y^* and F_z^* may be transformed:

$$\left.\begin{aligned} F_y^* &= \int_0^L ds \int_{-t/2}^{t/2} (w_C - r_{nC} n)(y + n_y)\, dn \\ &= F_y - \int_0^L ds \int_{-t/2}^{t/2} r_{nC} n^2 \cos \alpha\, dn \\ &= F_y - (t^3/12) \int_0^L r_{nC} \cos \alpha\, ds \end{aligned}\right\} \quad . \quad . \quad . \quad (A.61a)$$

since the other two terms of the product give zero when integrated between the limits $-t/2$ and $t/2$. The definition of F_y is given in Eq. A.20. Similarly, one may write

$$F_z^* = F_z - (t^3/12)\int_0^L r_{nC} \sin \alpha ds \quad . \quad . \quad . \quad (A.61b)$$

It should be remembered that in Eqs. A.60 and A.61 y and z are the co-ordinates of the median line measured from the centroid of the section. It may be seen that in the case of most thin-walled (t very small) sections the difference in the location of the centre of twist according to Eqs. A.59 and according to Eqs. A.22 is insignificant.

For the calculation of the shearing stresses associated with non-uniform warping one may assume that the component parallel to the median line is the same as that calculated in Sec. A.4 and given in Eq. A.24:—

$$\tau = -E\phi'''\int_0^s w_T ds \quad . \quad . \quad . \quad . \quad (A.24)$$

except that the warping w_T corresponds to twisting about the centre of twist as defined by Eqs. A.59 rather than by Eqs. A.22. It should be noted that w_T and not w_T^* figures in this equation. With this assumption the torque T of these shearing stresses with respect to the centre of twist is again given by

$$T = -E\phi'''t\int_0^L w_T^2 ds$$

On the other hand the shearing stresses according to Eq. A.24 have a non-vanishing resultant if the wall thickness of the section is considered finite. This is seen when one calculates the components of the resultant in the same way as in Sec. A.4. One obtains

$$S_y = E\phi'''t\left\{\left[-y\int_0^s w_T ds\right]_0^L + \int_0^L w_T y ds\right\}.$$

The first term in the twisted brackets vanishes for reasons stated in Sec. A.4. The second may be calculated from the first of Eqs. A.58 in conjunction with Eqs. A.55*a* and *b*. One has

$$\begin{aligned}\phi'' E\int_0^L ds\int_{-t/2}^{t/2}(w_T - r_{nT}n)(y + n_y)\,dn &= 0\\ &= t\int_0^L w_T y ds - \int_0^L ds\int_{-t/2}^{t/2} r_{nT}n^2\cos\alpha\, dn\\ &= t\int_0^L w_T y ds - (t^3/12)\int_0^L r_{nT}\cos\alpha ds\end{aligned}$$

Consequently

$$S_y = E\phi'''(t^3/12)\int_0^L r_{nT}\cos\alpha ds \quad . \quad . \quad . \quad . \quad (A.62a)$$

Similarly one obtains

$$S_z = E\phi'''(t^3/12)\int_0^L r_{nT}\sin\alpha ds \quad . \quad . \quad . \quad . \quad (A.62b)$$

The shearing stresses according to Eq. A.24 do not satisfy the equilibrium conditions of a small element cut out of the bar. The conditions may be satisfied, however, if one assumes that, simultaneously with the stresses according to Eq. A.24, shearing stresses τ_n are acting in the direction of the perpendicular to the median line. From the theory of elasticity one has

$$(\partial\sigma_x/\partial x)+(\partial\tau/\partial s)+(\partial\tau_n/\partial n)=0.$$

This equation may be transformed with the aid of Eqs. A.55*a*, A.55*b*, and A.24:—

$$\phi'''E\,(w_T-r_{nT}n)-\phi'''Ew_T+(\partial\tau_n/\partial n)=0.$$

Consequently,

$$\tau_n=\phi'''Er_{nT}\int_0^n n\,dn+f(x,s)$$

$$=\phi'''Er_{nT}\,(n^2/2)+f(x,s).$$

The unknown function $f(x, s)$ can be determined from the requirement that the shearing stress τ_n must vanish at the free surfaces $n=-t/2$ and $n=t/2$. Thus

$$\tau_n=(1/8)\,\phi'''Er_{nT}\,(4n^2-t^2) \quad . \quad . \quad . \quad . \quad (A.63)$$

The y-component R_y of the resultant of these shearing stresses is

$$R_y=\int_0^L ds\int_{-t/2}^{t/2}\tau_n\cos\alpha\,dn=-\phi'''E\,(t^3/12)\int_0^L r_{nT}\cos\alpha\,ds \quad . \quad . \quad (A.64a)$$

Similarly, the z-component R_z of the resultant is

$$R_z=-\phi'''E\,(t^3/12)\int_0^L r_{nT}\sin\alpha\,ds \quad . \quad . \quad . \quad (A.64b)$$

It may be seen that the resultant R according to Eq. A.64 just balances the resultant S according to Eqs. A.62. The moment T of the shear stresses τ_n with respect to the centre of twist is

$$T_n=\int_0^L ds\int_{-t/2}^{t/2}\tau_n r_{nT}\,dn$$

$$=-\phi'''E\,(t^3/12)\int_0^L r_{nT}^2\,ds.$$

This may be written in the form

$$T_n=-E\phi'''\Gamma_n \quad . \quad . \quad . \quad . \quad . \quad (A.65)$$

where

$$\Gamma_n=(t^3/12)\int_0^L r_{nT}^2\,ds \quad . \quad . \quad . \quad . \quad (A.65a)$$

Altogether the moment of all shearing stresses associated with non-uniform warping is

$$T_{tot}=-E\phi'''\,(\Gamma+\Gamma_n) \quad . \quad . \quad . \quad . \quad . \quad (A.66)$$

where Γ is the quantity defined in Eq. A.28. Since Γ is proportional to t, and Γ_n to t^3, the latter may be neglected for most thin-walled sections. Exceptions to this rule are, for instance, the angle- and T-sections, for which Γ vanishes.

A9. The Warping of Common Aircraft Sections.

In the following the warping w and the warping constant Γ (and/or Γ_n) are given for a few sections common in aircraft construction. The sections are

shown in Fig. A6. The co-ordinate s is measured along the median line of the section continuously from one free edge to the other, except in the case of the I- and T-sections, where such a definition cannot be maintained. The wall thickness t is assumed to be small and the warping across the thickness is

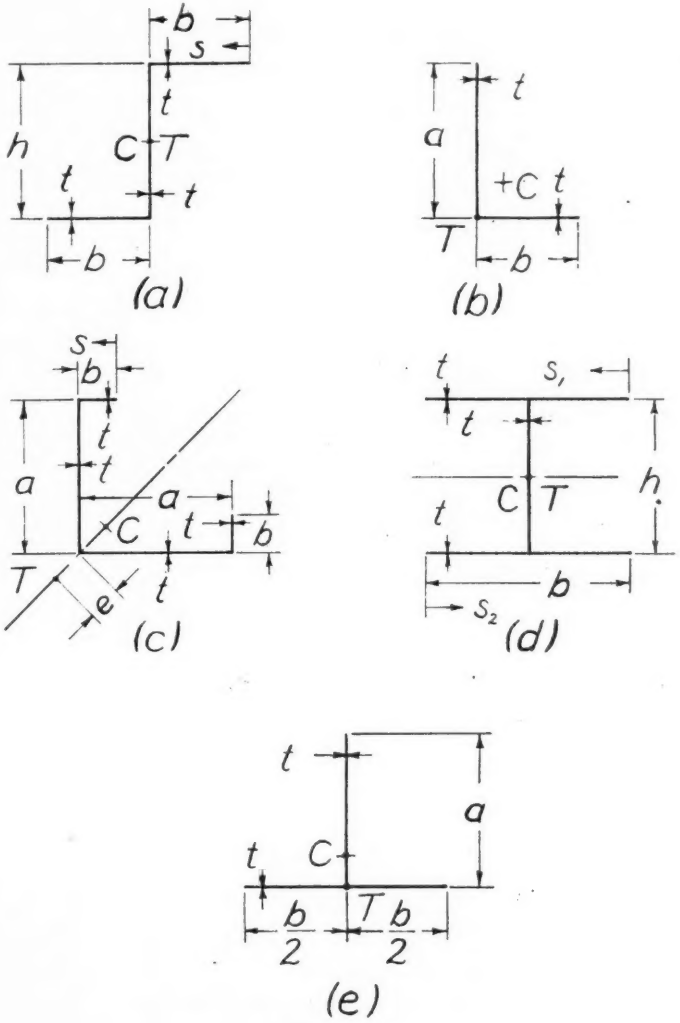

FIG. A6.

Common thin-walled sections.

disregarded, except in the cases of the angle- and T-sections, where Γ is zero for a rotation about the centre of twist.

All the formulæ given are valid for a rotation about the centre of twist. If the rotation is about some other axis, the value of w may be calculated with

the aid of Eq. A.16 from the formulæ given below. Eq. A.16 may also be used, together with Eqs. A.18*a* and A.28, to develop a formula for Γ_P for a rotation about any point P in the plane of the section :—

$$\Gamma_P = \Gamma_T - 2\,(y_P - y_T)\,(z_P - z_T)\,I_{yz} + (y_P - y_T)^2\,I_{zz} + (z_P - z_T)^2\,I_{yy} \quad . \qquad (A.67)$$

where Γ_T is the warping constant for a rotation about the centre of twist T, y_P and z_P are the rectangular co-ordinates of the point P, y_T and z_T those of the point T measured from the centroid C of the section, and the second moments of the section I_{yy}, I_{zz}, and I_{yz} are defined in Sec. A6.

A. *Channel Section.*—The pertinent formulæ are given in Sec. A7.

B. *Z-Section.*—The shear centre, or centre of twist, of the Z-section coincides with its centroid, and is in the middle of the side h in Fig. A.6*a*. If the section is twisted about its centroid, the warping is

$$\left.\begin{aligned} w &= -(h/2)\,b + C_1 \text{ when } b \leq s \leq (b+h) \text{ (web)} \\ w &= -(h/2)\,s + C_1 \text{ when } o \leq s \leq b \text{ (upper flange)} \\ w &= (h/2)\,(2b + h - s) + C_1 \text{ when } (b+h) \leq s \leq (2b+h) \text{ (lower flange)} \end{aligned}\right\} \quad (A.68)$$

The warping constant is

$$\Gamma = kth^5 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (A.69)$$

where

$$\left.\begin{aligned} k = (1/12)\,(b/h)^3\,[4\,(b/h)^3 + 12\,(b/h)^2 + 9\,(b/h) + 2]/[8\,(b/h)^3 \\ + 12\,(b/h)^2 + 6\,(b/h) + 1] \end{aligned}\right\} \quad (A.69a)$$

C. *Angle Section.*—The shear centre, and consequently the centre of twist, is the intersection point of the median lines of sides a and b in Fig. A.6*b*. The warping w is, therefore, zero everywhere, and so is Γ. The warping w_n, however, does not vanish, and the warping constant

$$\Gamma_n = (t^3/36)\,(a^3 + b^3) \quad . \quad . \quad . \quad . \qquad (A.70)$$

D. *Flanged Angle Section.*—The centre of twist T of the flanged angle section, shown in Fig. A.6*c* is on the axis of symmetry of the section at a distance e from the edge of the section :—

$$e = (a/\sqrt{2})\,(b/a)^2\,[3 - 2\,(b/a)]/[1 + 3\,(b/a) - 3\,(b/a)^2 + (b/a)^3] \qquad (A.71)$$

The warping of the section is given by the equations

$$\left.\begin{aligned} w &= [a + (e/\sqrt{2})]\,(b - s) - (e/\sqrt{2})\,a \text{ when } o \leq s \leq b \\ w &= (e/\sqrt{2})\,(s - a - b) \text{ when } b \leq s \leq 2a + b \\ w &= [a + (e/\sqrt{2})]\,(2a + b - s) + (e/\sqrt{2})\,a \text{ when } 2a + b \leq s \leq 2a + 2b \end{aligned}\right\} \quad (A.72)$$

The warping constant is given by the equation

$$\Gamma = kta^5 \quad . \quad . \quad . \quad . \quad . \qquad (A.73)$$

where

$$k = (1/6)\,(b/a)^3\,[4 + 3\,(b/a)]/[1 + 3\,(b/a) - 3\,(b/a)^2 + (b/a)^3] \qquad (A.73a)$$

E. *I-Section.*—In the case of the I-section the co-ordinate s_1 is measured along the upper flange from $s_1 = o$ to $s_1 = b$; the co-ordinate s_2 is measured along the lower flange from $s_2 = o$ to $s_2 = b$. Both are shown in Fig. A6*d*. The centre of twist coincides with the centroid. The warping is given by the equations

$$\left.\begin{aligned} w &= (1/4)\,h\,(b - 2s_1) \text{ upper flange} \\ w &= o \text{ web} \\ w &= (1/4)\,h\,(b - 2s_2) \text{ lower flange} \end{aligned}\right\} \quad . \quad . \quad . \qquad (A.74)$$

The warping constant is

$$\Gamma = (1/24)\,(b/h)^3\,th^5 \quad . \quad . \quad . \quad . \qquad (A.75)$$

F. *T-Section.*—The case of the T-section is similar to that of the angle section. With the notation of Fig. A6*e* the warping constant is

$$\Gamma_n = (t^3/36)\,[a^3 + (1/4)\,b^3] \quad . \quad . \quad . \quad . \qquad (A.76)$$

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REVIEWS.

Sub-Atomic Physics.

H. Dingle, Nelson's Aeroscience Manual. T. Nelson and Sons, Ltd. 1942. 5/- net.

This book forms Volume II of a course of Physics for Aeronautical Students of which Volume I on Mechanical Physics has already been reviewed in the Journal.

Like its predecessors in this excellent series, this is a first-class book. It covers in an introductory chapter the structure of the atom, and in the remainder of the book optics and electricity and magnetism. Professor Dingle not only writes well, but clothes the most difficult part of the book in common place analogy and language which go so far to make the student not only understand what the author is writing about, but also make him interested in what he is reading; and interest is ninety per cent. of the battle of educating. Nor does Professor Dingle shirk the absence or inadequacy of an explanation by a tangle of verbiage. Having spent over twenty pages describing in some detail the structure of the atom he adds:

"The picture of the atom and its behaviour which we have given is the one which enables us to explain the largest number of facts without becoming too abstruse. The latest theories explain more but are not so easily grasped, present no imaginable picture of the atom, and are, moreover, liable to supersession by still more comprehensive theories as progress continues. The reader must understand that we do not say the atom is exactly as we have pictured it; it appears, in fact, that a precise picture is impossible to give. The value of the view we have adopted is that it gives a satisfactory and not misleading account of the facts which concern us in practical work. Where it fails we shall simply describe the facts without attempting to explain them."

Each chapter is followed by a series of useful exercises for the student. The book is excellently printed and illustrated and can be thoroughly recommended to all students taking the subject of physics in their examinations, whether aeronautical or not.

Air Navigation: British Empire Edition (2nd Ed.).

By P. V. H. Weems. McGraw Hill. 1942. 35/-.

Any comment upon this work would be superfluous, since it has for many years been established as a classic. The durable value of its contents is emphasised by the fact that practically nothing in the way of theory, methods or examples has been altered from the first edition. The changes brought about by five years of progress mainly concern improved types of navigation instruments. We have, for example, a description and photograph of the Astro Compass (pp. 99-104); Drift Recorder, Mk. II; Course and Speed Calculator, Marks II and II*a*; Navigational Computer, Mark III; and in a later chapter a description of the R.A.F. Bubble Sextant, with a much fuller account of methods of observation by Bubble Sextants and the errors to which they are liable. Another useful addition is the method of finding fixed objects by Sun position lines (p. 366).

To make room for this additional matter, very little has been omitted. Chapter XX, dealing with Navigational Equipment, has been entirely left out, presumably as being a mere catalogue of instruments that are described elsewhere; while a further three pages have been saved by cutting out some notes on outstanding navigation flights, which, though interesting in themselves, are now ancient history. A minor, but significant alteration is the omission of the title "Colonel" before Lindbergh's name—*sic transit gloria mundi*!

ASTRONOMICAL NAVIGATION.

By W. M. Smart. Longmans, Green and Co. 1942. 5/-.

This is the promised sequel to Professor Smart's " Introduction to Sea and Air Navigation " and should be read in conjunction with it. The subject is handled in the lucid and practical manner which we have come to associate with the author. After a preliminary chapter giving the essential definitions and formulæ for spherical triangles, the book goes on to deal with the Celestial Sphere, the Astronomical Triangle, Greenwich Mean Time, Altitude Observations, the Astronomical Position Line, and the Air Astronomical Navigation Tables, while a concluding chapter is mainly concerned with Altitude Curves and other short-cut methods used by practising navigators. An Appendix reproduces some pages from the " Air Almanac " and the " Astronomical Navigation Tables." Each chapter contains a series of Exercises, to which answers are given at the end of the book. Star charts of the Northern and the Southern Heavens are printed, respectively, on the fly-leaves at the beginning and end. No space has been wasted, but the quality of paper and printing is much better than in many wartime editions. In general, this is a book which should certainly be studied by all student navigators and might well be read as a " refresher course " by those who are more advanced.

FLIGHT HANDBOOK. A GUIDE TO AERONAUTICS, AIRCRAFT ENGINES AND SUPERCHARGERS. (3rd Edition.)

Flight Publishing Co., Ltd. 1942. 6/- net.

The third edition of this most useful handbook has been considerably enlarged and a new section added on typical aircraft power units and supercharging. Many new illustrations, too, have been added, bringing the book completely up-to-date.

There is not much doubt " Flight Handbook " will grow larger and more popular with each issue. For anyone with no knowledge of aeronautics, and eager to learn, this book will give him all he wants, in clear language, from the aerodynamics of the aeroplane to an explanation of the working principles of the instruments which are of such vital aid to the pilot. The new engine section is particularly well-illustrated with cut-away diagrams which mean so much to those who are becoming acquainted for the first time with engine construction.

GAS TURBINES AND JET PROPULSION FOR AIRCRAFT.

G. Geoffrey Smith, M.B.E. Flight Publishing Co., Ltd. 1943. 3/6.

This is an important book. Mr. Geoffrey Smith has made a very thorough search for the work which has been done on gas turbines and jet propulsion for aircraft throughout the world. With great skill he has written his book in as simple terms as possible and has given to the young aircraft engineers, in whose hands the future of aircraft design lies, enough food for thought to keep them healthy for many years to come.

" The idea of an aircraft being propelled without the aid of an airscrew strikes one at first as fantastic," says the author. It might have the advantage of settling once and for all whether the term propeller or airscrew should be used. There is little doubt, however, that jet reaction has come to stay in some form or other and that it will play a major part in powering air transport of the future.

The advantages of jet propulsion units or reciprocating engine units are such that intensive research and development are bound to be encouraged by the governments of the world to bring about the practical results which appear possible.

With jet propulsion safety fuels can be used with direct application of the power produced. Construction of jet propulsion units is lighter and the power-weight ratio better than those of the reciprocating engine. The abolition of the airscrew will lead to lighter undercarriages, and units can be completely enclosed in the wings and fuselage, with an increase in aerodynamic efficiency.

There are disadvantages, but most of these are those of development, and no serious disadvantage from a theoretical point of view presents itself. Jet propulsion, as the author rightly points out, is particularly suitable for aircraft operating at high-altitude levels, a fact of some importance for the progress of civil aviation.

The coming of the jet will bring about new ideas in design, and aircraft of the future will make some of the best aircraft of to-day appear like ugly ducklings. A new era in power for the aeroplane is dawning. That does not mean to say the reciprocating engine is dead, or even dying, but it does mean that undoubtedly for the large, long-range, high-speed aircraft of the future jet propulsion offers more possibilities than the aircraft driven by airscrews.

Mr. Geoffrey Smith has done well to collect together existing information and to point out the possible future of the jet. In this Journal was published the first article written by Whittle, whose name is so frequently mentioned in Mr. Geoffrey Smith's pages, as long ago as 1931, on the turbo compressor.

Germany and Italy have paid special attention to jet propulsion and its possibilities. The only thing which puzzles the reviewer is why Italy gave so much publicity to the Caproni-Campini on its flight from Milan to Rome in November, 1941.

AIR PILOT TRAINING.

Bert A. Shields. McGraw Hill Book Co., London and New York. 1942. 24/- net.

This book is essentially intended to provide, as the author states in his preface, " in a single volume all the information necessary to pass the written examinations for a private and a commercial pilot licence." The licences are American licences and the whole book is written on the basis of American practice.

There are four sections, the first on aircraft and theory of flight, the second on aircraft engines, the third on meteorology, and the last on air navigation.

Mr. Shields is himself a licensed pilot with over 5,000 hours flying. He is also a licensed aeroplane and engine mechanic and an instructor in the subjects he has covered in his book. He particularly stresses that " great pains have been taken to avoid all unnecessary technical terms."

The author has, indeed, gone out of his way to explain in straightforward, simple language his subjects, but it does not always follow that such language is the clearest and most accurate for examination purposes. The reviewer wonders what marks an examiner would give to the candidate who was asked " What is gravity? " and receives the reply " Gravity is the force that gave early pioneers their greatest difficulty."

But these things are a matter of teaching. There is much in this book which will prove useful and informative to those wishing to acquire a knowledge of the why and wherefore of flight. It has the merit, too, of being excellently printed and illustrated.

THE AVENGERS.

By Charles Graves. Hutchinson. 1942. 7/6.

SEVEN PILOTS.

By Charles Graves. Hutchinson. 1943. 8/6.

These two novels are, respectively, the second and third in the series initiated by " The Thin Blue Line." They belong to that curious type of fiction which used to be called " roman à clef," for which the recipe is simple: take a real man named (say) Donald Black, call him " Ronald White," describe in detail some flying adventure of his, altering the time, place, and possibly the type of his machine, and insert somewhere in the book an official photograph taken of some similar incident. The result is no doubt highly diverting to pilots, and those in their immediate circle who can identify the originals of the characters and incidents described, and to all those youngsters whose ambition is to join the

R.A.F. The latter, when they have followed Mr. Graves's heroes through three books, will certainly be equipped with an extensive vocabulary. Personally, the reviewer found some difficulty in distinguishing the seven (to him) quite uncharacterised heroes, and could have put either book down at any point without any urge to find out what happened to them; but, since 30,000 copies of "The Thin Blue Line" and 20,000 of "The Avengers" have been sold, it appears that he is in a minority.

PLANES EXPLAINED.

Roger Tennant. Argus Press. 1942. 1/- net.

This is a book which is written to appeal to that ever-increasing public which wants to know why aeroplanes, apparently called upon to do the same job, have such different shapes and characteristics. Here one learns why wings are elliptical, tapered, thick or thin, and so on; why there are low-winged, mid-winged and high-winged monoplanes; why some aircraft have one rudder and others two; why some under-carriages are fixed and others retractable; and so on. The book is illustrated by just those examples which will be of such use to pilots, to members of the Royal Observer Corps and others who are very much concerned with identifying aircraft. There is not the slightest doubt if they know the why of a particular shape they will remember it far more easily than if they have a little more to go on than that it is so.

This is a most useful, interesting and informative shillingsworth.

ADVANCED FLYING.

Captain Norman Macmillan, M.C., A.F.C. George Allen and Unwin, Ltd. 1943. 2/6 net.

Captain Norman Macmillan is not only a good pilot but he is also a good writer, two things which a spate of war books shows do not often go together. Captain Macmillan, indeed, has a long and growing list of books in his name, most of them worth while reading many times. His wide experience from fighting in the last war to test pilot and flying in many of the different parts of the world on many types of aircraft, enables him to write with some authority.

In his first chapter he deals with the flying of fighters and the overwhelming importance of the care of his eyesight to the fighter pilot. Captain Macmillan points out the great advantages the modern fighter has over the fighter in the last war, but rightly insists that in the ultimate fight it is the pilot who sees more in less time who stands the best chance.

The difference between flying multi-engined aircraft and single-engined are clearly explained in the second chapter, one which is naturally followed by a chapter on flying on one engine. Captain Macmillan's description of the return of a Hampden from Wilhelmshaven on one engine is not only first-class, but will give many a newly-fledged pilot confidence when forced to fly under similar conditions.

In Chapter IV the author explains what is meant by the safety factors of an aeroplane, pointing out how worth while it is to know the strength of the machine you are flying, "and helps you to decide the problem of whether to bale out or to fly the crate home like 'a piece of lace.'" The Forces in Flight, the subject of Chapter V, naturally follows. Chapter VI is concerned with the effect of manœuvres both on the aeroplane and the pilot.

Chapter VII covers Taking-off. As the author remarks, "Every take-off varies." Captain Macmillan explains how certain games improvea pilot's take-off abilities, and the reviewer heartily agrees with his comment. "The static-ball game of golf is not a pilot's game." Chapter VIII is concerned with the causes and dangers by the stall, and the last chapter with landings and under-carriages.

This is a very good half-a-crown's worth and even the most experienced pilot will get his half-a-crown back in some useful piece of knowledge.

AN INTRODUCTION TO THE STUDY OF MAP PROJECTION.

By James Mainwaring. Macmillan and Co. 1943. 5/-.

It is several years since there has been published a good elementary text-book on map projection. The present work is therefore particularly welcome, at a time when so many young men are studying navigation in the A.T.C. or pre-entry classes for the R.A.F. It is divided into three parts, entitled respectively: The Problem of Map Projection, The Process of Map Projection, and The Projections Applied. Of these, the second is, of course, the fullest and most important. Cylindrical, conical and zenithal projections are dealt with in turn, with a profusion of excellent diagrams and the strict minimum of essential mathematics. Particularly valuable are the comparative diagrams (Nos. 25, 26 and 27) summing up the scale differences between the various types of projection. Diagram 44 and the hints on the page facing it will also be of great use to the student who may be called upon to recognise any given projection in an examination. The book includes a series of test questions which the student may work out for himself from the text, or with the help of a teacher (answers are given only in a few numerical cases).

COASTAL COMMAND AT WAR.

By " Squadron-Leader Tom Dudley Gordon." Jarrolds, 1943. 7/6.

By a coincidence this book appears simultaneously with the Air Ministry official publication entitled " Coastal Command." Many of the illustrations are the same—or obviously taken on the same occasion—and very beautiful they are. For sheer technical perfection it would be hard to beat the aerial views of the Norwegian fiords, the treeless hills of Iceland, or the floating pack-ice. The reader will be well advised first to run through the Air Ministry booklet, then to re-read the same incidents in the present book, where he will find the "human interest " angle which is necessarily left out of the official account. From internal evidence, the author is a professional writer: he has a dry and witty style, a nice sense of characterisation, and the ability to tell a good story. We particularly enjoyed his anecdotes of the Rear-Admiral Pilot Officer (page 31); of the " mis-spelled pleasantry " from the Hudson navigator (page 36), and of the German commander's signal: " What *is* ' scuttle ' please? " On the more serious side, we have vivid descriptions of the Altmark reconnaissance, the work of Coastal Command over Dunkirk, and its crowning exploit in hunting the Bismarck.

ELEMENTARY ELECTRICITY FOR RADIO STUDENTS.

By W. E. Flood. Longmans Green. 1942. 1/-.

This addition to the "Service Manuals " series will be welcomed by A.T.C. cadets and prospective wireless operators. Most text-books on electricity, even those describing themselves as " elementary," assume that the reader knows at least the ABC of the subject, and this frequently means that he is confronted with terms, quantities and circuit drawings which he finds puzzling. The present booklet assumes zero knowledge, and proceeds to impart the ABC in a pleasant and lucid way. Every symbol is identified, every term explained in detail, and every explanation is followed by a simple numerical example, worked out in full. Although the book is only 62 pages long, it contains an index, exercises, and a set of revision problems.

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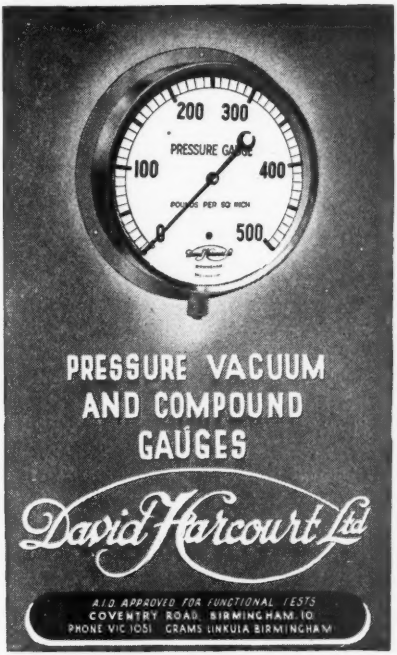

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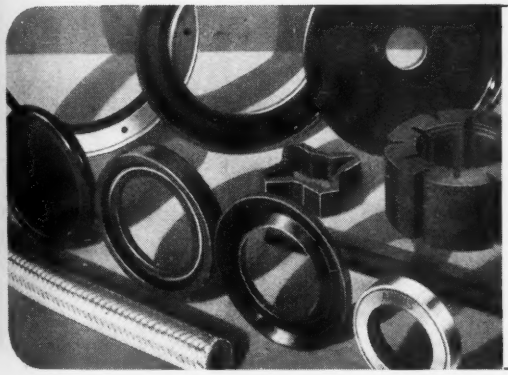

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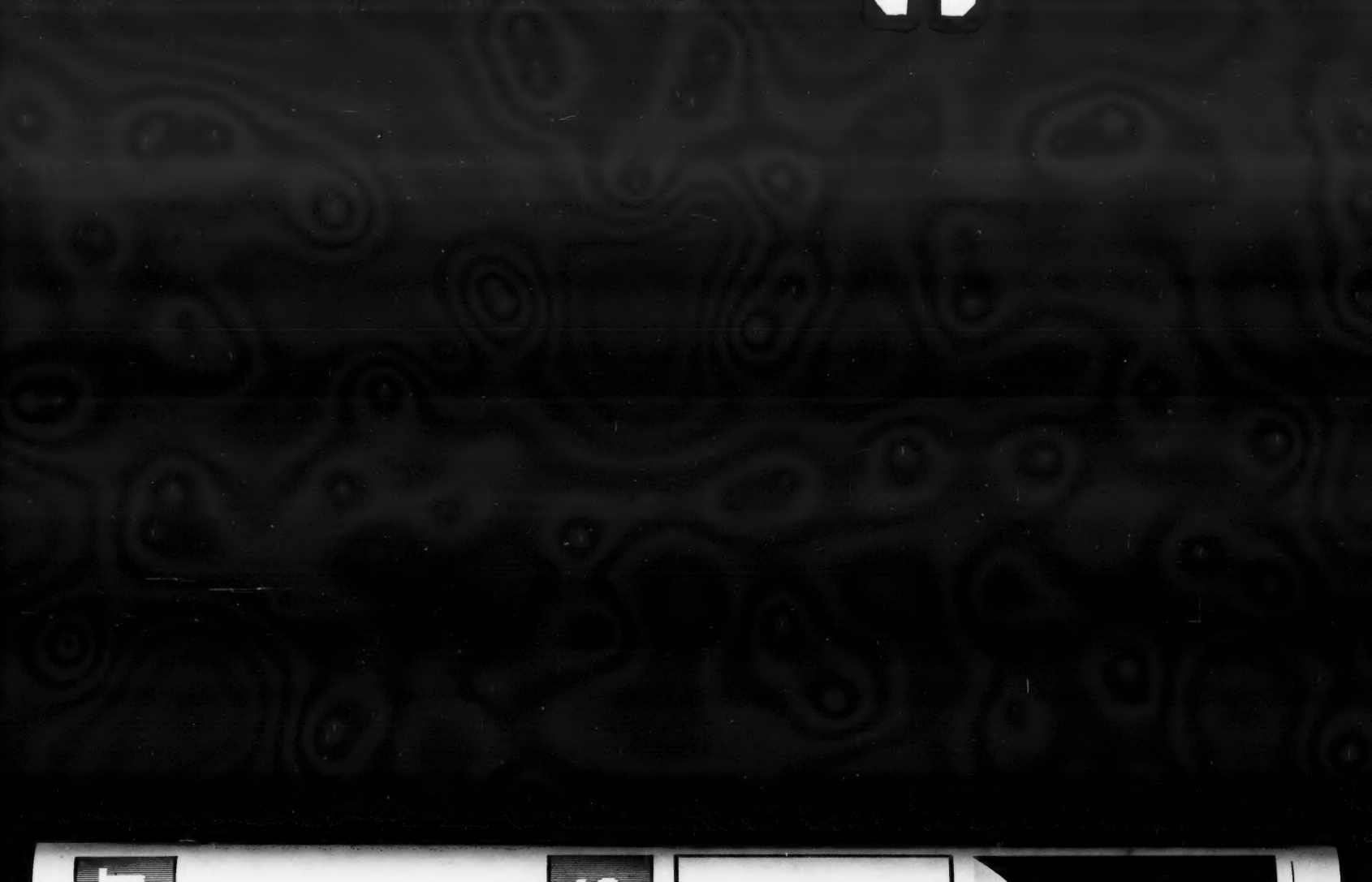
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